SACE BOARD
SOUTH AUSTRALIA

Stage 2 Mathematical Methods

Sample examination questions - 1

Government
of South Australia

Question 1 (7 marks)

(a) For the functions below, determine $\frac{dy}{dx}$. You do not need to simplify your answers.

(i) $y = 4x^3 \sin 2x$.

(3 marks)

(ii) $y = \sqrt{4 - x^2}$.

(2 marks)

(b) Find $\int e^{2x+1}\,\mathrm{d}x$.

(2 marks)

Question 2 (6 marks)

Consider the function $f(x) = \ln x + \ln(x+4)$.

(a) For what values of x is $f(x)$ defined?

(1 mark)

(b) Show that $x^2 + 4x - e = 0$ when $f(x) = 1$.

(2 marks)

(c) Using the information in parts (a) and (b), find the *exact* solution(s) to the equation $\ln x + \ln(x+4) = 1$.

(3 marks)

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Question 3 (7 marks)

Consider the function $f\left(x\right)=\frac{\sqrt{x}}{\left(5-x\right)^{2}}$.

(a) Show that $f'\left(x\right)=\frac{3x+5}{2\sqrt{x}\left(5-x\right)^{3}}$.

(4 marks)

(b) Find the equation of the tangent to the graph of $f\left(x\right)$ at $x=4$. Write your answer in the form $y=mx+c$.

(3 marks)

Question 4 (7 marks)

(a) Find $\int 6\sin\frac{x}{2}\,\mathrm{d}x$.

(2 marks)

(b) Hence show that $\int_{0}^{2\pi} 6\sin\frac{x}{2}\,\mathrm{d}x = 24$.

(2 marks)

(c) Find the value(s) of a for $0 \le a \le 2\pi$, such that $\int_0^a 6\sin\frac{x}{2}\,dx = 6$.

(3 marks)

Question 5 (7 marks)

The discrete random variable X can take the value 0, 1, 2, 3, 4, or 5.

(a) The following information is known:

$$\Pr(X=0)=\Pr(X=1)=\Pr(X=2)=\Pr(X=3)=m$$

and

$$\Pr(X=4)=\Pr(X=5)=n.$$

(i) Show that $4m+2n=1$.

(1 mark)

(ii) Find $\Pr(X \leq 1)$, in terms of m.

(1 mark)

Consider the following statement:

$$\Pr(X \geq 2)=4\times\Pr(X \leq 1).$$

(b) Using the information provided in part (a), and the results from parts (a)(i) and (a)(ii), show that the only possible solutions to this statement are $m=\frac{1}{10}$ and $n=\frac{3}{10}$.

(3 marks)

(c) Calculate μ_X.

(2 marks)

Question 6 (6 marks)

Many people are members of social media sites such as WeChat, Facebook, and Instagram.

Early research shows that 64.1% of social media users have, at some stage, taken a long-term break from using social media.

In 2018, a campaign encouraged people to take a break from using social media. Subsequently, researchers conducted a survey to determine if this campaign had affected people's use of social media. In this survey, the researchers interviewed 949 social media members and found that 635 had recently taken a break from using social media.

(a) (i) Calculate a 95% confidence interval for the proportion of the population of social media users who have taken a break from using social media.

(2 marks)

(ii) Based upon the confidence interval that you calculated in part (a)(i), can the researchers claim that a greater proportion of social media users have taken a break from using social media after the campaign, compared with before the campaign? Justify your answer.

(2 marks)

(b) The researchers would like to increase the accuracy of their findings by reducing the width of the confidence interval used.

Which *one* of the following methods is appropriate for the researchers to use? Tick the appropriate box, and justify your answer.

☐ Increase the sample size.

☐ Change the confidence level.

Justification:

(2 marks)

Question 7 (8 marks)

Consider the function $f(x)=\dfrac{4}{x+1}$.

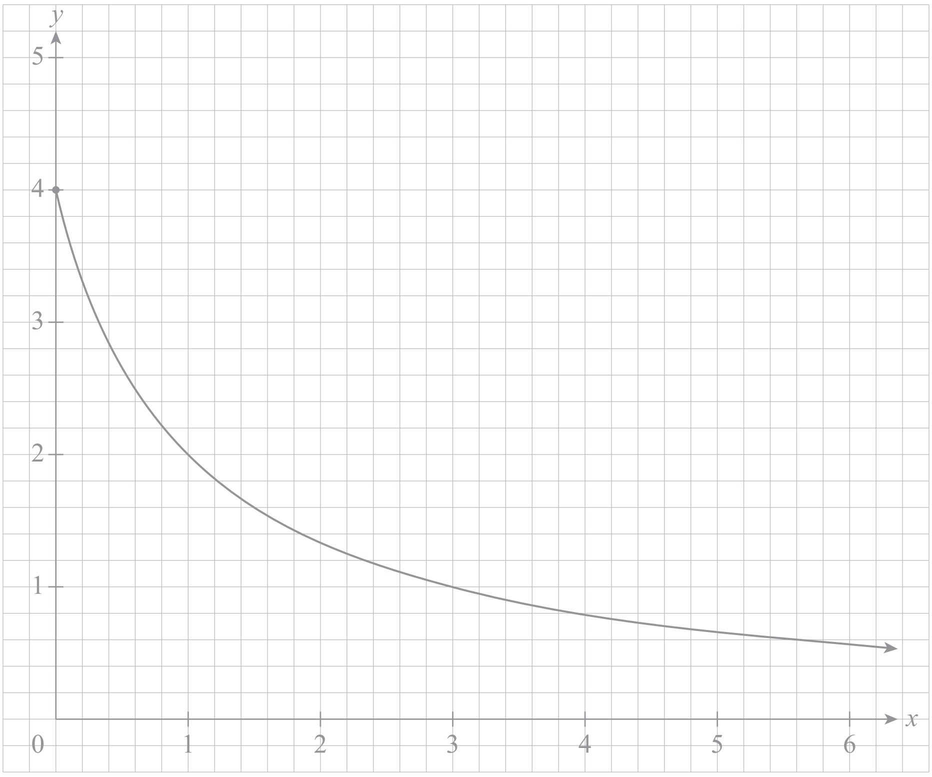
A graph of $y=f(x)$ is shown below for $x \geq 0$.

(a) (i) On the graph above, draw *two* rectangles of equal width that represent an underestimate for the value of $\displaystyle\int_0^4 \frac{4}{x+1}\,\mathrm{d}x$. (1 mark)

(ii) Calculate the *exact* value of this underestimate.

(2 marks)

You may use the spare graph provided below when answering part (b); however, you will not earn any marks by doing so.



(b) Calculate an improved underestimate for the value of $\int_0^4 \frac{4}{x+1}\,dx$, using *four* rectangles of equal width.

(2 marks)

(c) Find the *exact* value of $\int_0^4 \frac{4}{x+1}\,\mathrm{d}x$.

(3 marks)

Question 8 (6 marks)

The process of buying a train ticket from a ticket machine is a 'transaction', and the time taken to complete a transaction is the 'transaction time'. At a particular machine, transaction times are known to be normally distributed, with a mean of 45 seconds and a standard deviation of 20 seconds.

A survey is carried out to review the transaction times at this machine. Thirteen transaction times are randomly selected from the population of transaction times at this machine.

(a) Let S_{13} be a random variable defined as the sum of 13 randomly selected transaction times.

(i) (1) What is the shape of the distribution of S_{13}?

(1 mark)

(2) Explain your answer to part (a)(i)(1).

(1 mark)

(ii) What is the mean of S_{13}?

(1 mark)

(iii) What is the standard deviation of S_{13}?

(1 mark)

(b) There are 12 people waiting in a queue to complete their transactions at the ticket machine. Lisa joins the end of this queue.

Lisa needs to complete her transaction within 10 minutes in order to catch her train.

What is the probability that Lisa will complete her transaction in time to catch her train?

(2 marks)

Question 9 (9 marks)

(a) Show, by differentiation, that $\int \ln(x-1)\,\mathrm{d}x = (x-1)\ln(x-1) - x + c$.

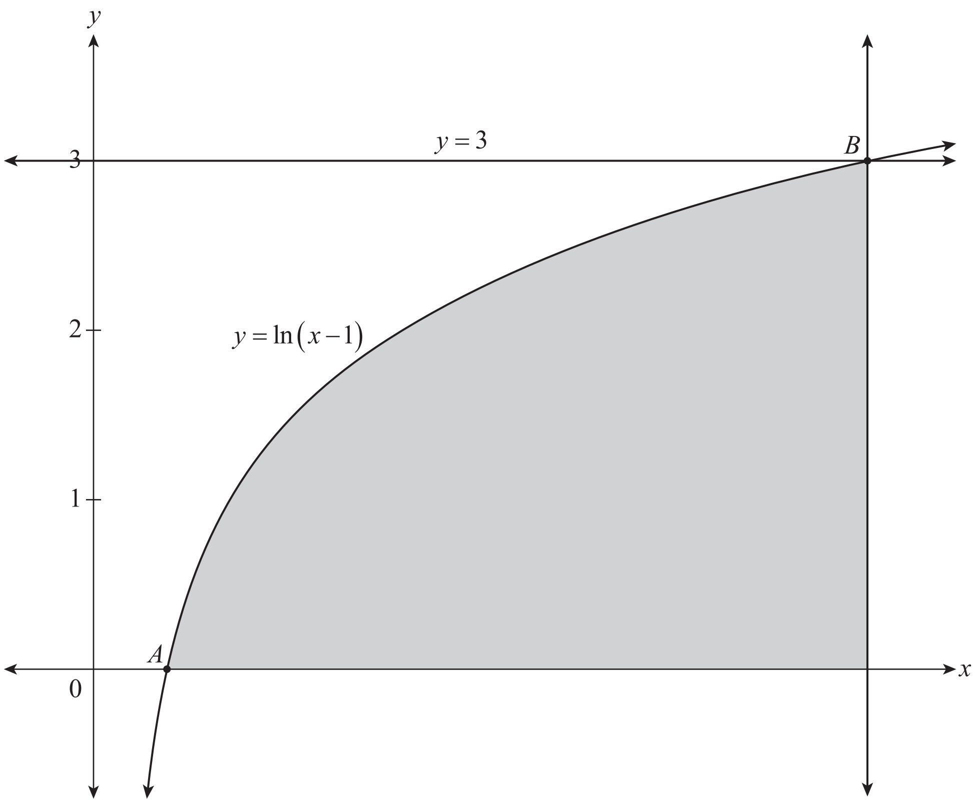
(2 marks)

Consider the function $f(x) = \ln(x-1)$.

The graph of $y = f(x)$ for $x > 1$ is shown below, together with the horizontal line $y = 3$.

The x-intercept is labelled as point A, and the point at which $f(x) = 3$ is labelled as B.

A vertical line has also been added, passing through B.



(b) What are the *exact* coordinates of A?

(2 marks)

(c) What are the *exact* coordinates of B?

(2 marks)

(d) Using the information given in part (a), and your answers to parts (b) and (c), find the *exact* value of the shaded area on the graph on page 22.

(3 marks)

Question 10 (9 marks)

Suppose that the amount of sleep time per night for secondary-school students is normally distributed, with a mean of 7.6 hours and a standard deviation of 1.24 hours.

Research suggests that secondary-school students should get at least 8 hours of sleep per night.

(a) Based on this distribution, what proportion of secondary-school students *do not* get enough sleep?

(2 marks)

A principal conducted a survey on the amount of sleep time per night of 200 randomly selected students in her secondary school, and found that the sample mean was 7.8 hours.

(b) Assume that the population standard deviation for the amount of sleep time per night is 1.24 hours.

Using the sample data, calculate a 95% confidence interval for the mean amount of sleep time per night of these secondary-school students.

(2 marks)

The principal issues a note to the teachers at her school, claiming that, on average, the school's students are not getting enough sleep.

(c) Does the confidence interval that you calculated in part (b) support the principal's claim? Justify your answer.

(2 marks)

(d) The principal claims that between 51% and 62% of the school's students are not getting enough sleep.

Provide mathematical calculations that support or contradict this claim.

(3 marks)

Question 11 (6 marks)

Consider the function $f(x)$, defined for $x \geq 0$.

The following information is given:

Domain	$f(x)$	$f'(x)$	$f''(x)$
$x=0$	2		
$0 \leq x < 3$		<0	>0
$x=3$	-2	0	0
$x>3$		<0	<0

(a) The graph of $y=f(x)$ has a stationary point of inflection.

What are the coordinates of this stationary point of inflection?

(2 marks)

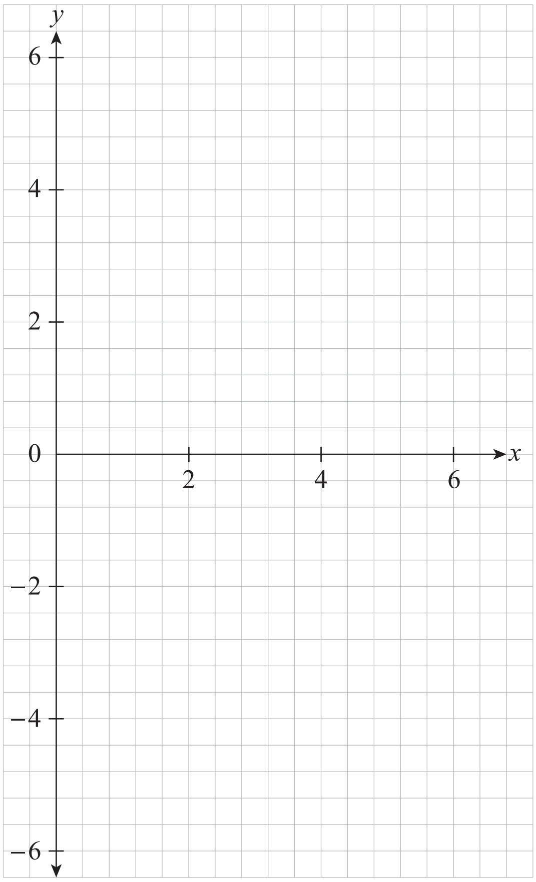
(b) Select the correct answer by ticking the appropriate box.

For $0 \leq x < 3$, the shape of the curve of $y=f(x)$ is:

☐ concave upwards (convex)

☐ concave downwards. (1 mark)

(c) On the axes below, sketch a continuous smooth curve of the graph of $y = f(x)$ for $x \geq 0$, using the given information and your answers to parts (a) and (b).



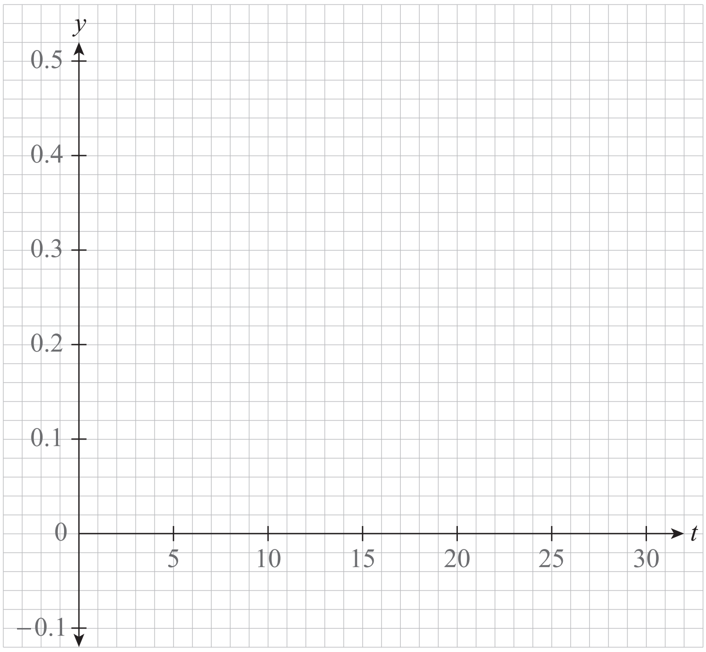
(3 marks)

Question 12 (11 marks)

When a person smokes a cigarette, they absorb into their bloodstream many toxic chemical compounds, including cyanide.

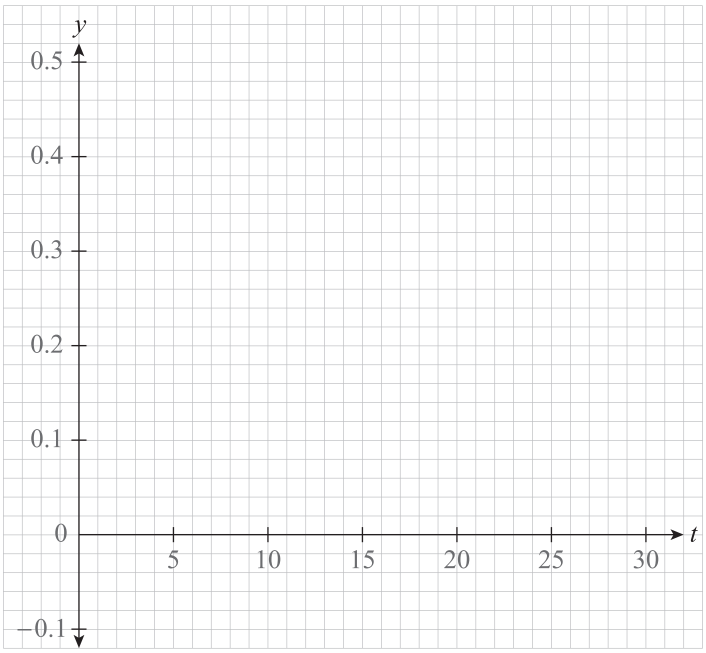
The function $C(t)=0.1+0.3t^{0.6}e^{-0.17t}$ can be used to model the blood cyanide concentration in micrograms per millilitre (µg/mL), t minutes after a person smokes a cigarette and does not smoke another cigarette.

(a) On the axes below, sketch a graph of $y=C(t)$.



(3 marks)

(b) On the axes below, sketch $y=C'(t)$.



(2 marks)

(c) (i) Interpret the significance of the value of 0.1 in the function $C(t)$.

(1 mark)

(ii) Interpret $C'(t)$ in the context of smoking a cigarette.

(1 mark)

(d) A person smokes a cigarette.

(i) Use the model to predict:

(1) the number of minutes for which the person's blood cyanide concentration continues to increase

(1 mark)

(2) the person's maximum blood cyanide concentration.

(1 mark)

(ii) Consider the point at which the person's blood cyanide concentration is decreasing at the greatest rate.

(1) At what time will this occur?

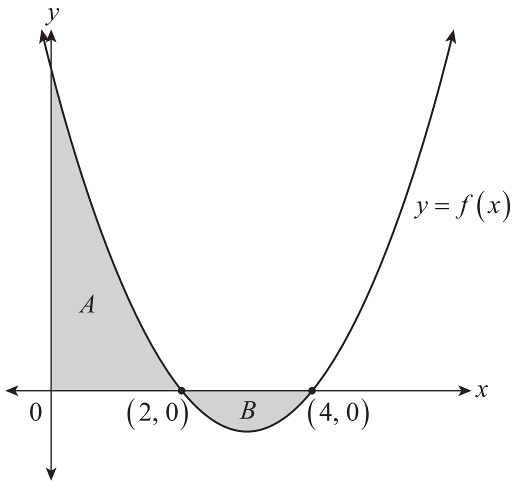
(1 mark)

(2) What will be the person's blood cyanide concentration at the time that you identified in part (d)(ii)(1)?

(1 mark)

Question 13 (8 marks)

Consider the graph of the function $y = f(x)$ shown below, where $f(x)$ is a quadratic.
The function $f(x)$ has x-intercepts at 2 and 4.
A is the area of the region bounded by the y-axis, the x-axis, and $y = f(x)$ for $0 \le x \le 2$.
B is the area of the region bounded by the x-axis and $y = f(x)$ for $2 \le x \le 4$.



(a) Given that $\int_0^4 f(x)\,\mathrm{d}x = 32$:

(i) Select the correct expression by ticking the appropriate box.

☐ $A + B = 32$

☐ $A + B < 32$

☐ $A - B = 32$

☐ $A - B > 32$ (1 mark)

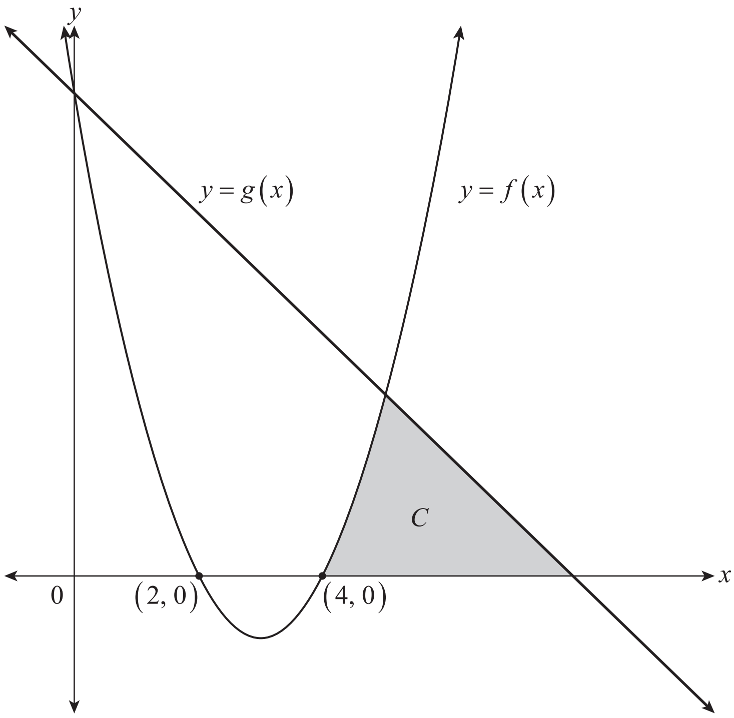
(ii) If A is 40 square units, find $\int_2^4 f(x)\,\mathrm{d}x$.

(1 mark)

Shown below is a general case of the graph of $y = f(x)$, for $f(x) = a(x-2)(x-4)$ where a is a real and positive constant.

On the same axes, the graph of the function $y = g(x)$ has also been drawn, where $g(x) = a(8-x)$.

C is the area of the region bounded by $y = f(x)$, $y = g(x)$, and the x-axis for $x \geq 4$.



(b) Show that $f(x)$ intersects $g(x)$ when $x = 0$ and when $x = 5$.

(2 marks)

(c) Find C in terms of a, giving your answer in its simplest form.

(4 marks)

Question 14 (10 marks)

Consider a multiple-choice question that contains five possible answers from which to choose. Only one of the five answers is correct.

(a) A student attempts this multiple-choice question. Let $X=1$ if the student randomly chooses the correct answer and $X=0$ if the student randomly chooses an incorrect answer.

(i) Construct a table of the probability distribution of X.

(2 marks)

(ii) Calculate the mean of X.

(1 mark)

(b) A student attempts a test containing 50 of these multiple-choice questions. In order to pass the test, the student must correctly answer at least 25 of the questions.

(i) The student knows the correct answer to 15 questions, and chooses these answers. The student randomly chooses an answer for each of the remaining 35 questions.

(1) What is the probability that the student passes the test?

(2 marks)

(2) In total, for this student, what is the expected number of correctly answered questions?

(2 marks)

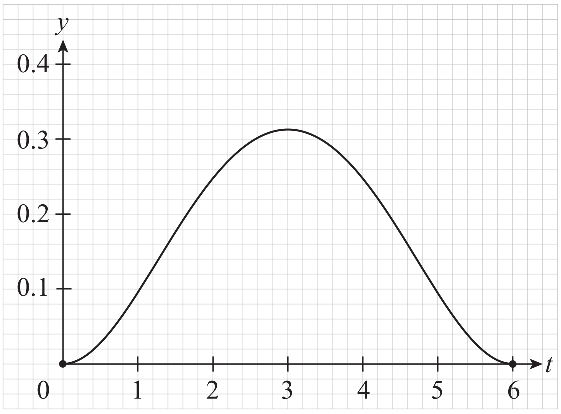
(ii) To have a probability of at least 33% of passing the test, what is the minimum number of questions to which the student must know the correct answer, given that the student will randomly guess the answers to the remaining questions?

(3 marks)

Question 15 (16 marks)

The probability density function $f(t) = at^2(6-t)^2$, where $0 \le t \le 6$, can be used to determine the probability that a particular type of mobile (cell) phone, called the 'Z', will be replaced at time t years after it was purchased.

The graph of $y = f(t)$ is shown below.



(a) Show that a must equal $\dfrac{5}{1296}$ in order for $f(t)$ to be a valid probability density function.

(3 marks)

(b) (i) Let the random variable T represent the time at which a randomly chosen Z phone will be replaced.

Find $\Pr(T > 2.8)$.

(2 marks)

(ii) (1) Write down an integral expression for μ_T, the expected time at which a Z phone will be replaced.

(1 mark)

(2) Hence or otherwise, find μ_T.

(1 mark)

(iii) Calculate σ_T, the standard deviation of the distribution of T values.

(2 marks)

(iv) A sample is taken of 100 randomly chosen people who have previously owned a Z phone. The mean of the times at which their phones were replaced, $\overline{T}$, is found.

(1) What is the distribution of $\overline{T}$?

(2 marks)

(2) Find $\Pr\left(\overline{T} > 2.8\right)$.

(1 mark)

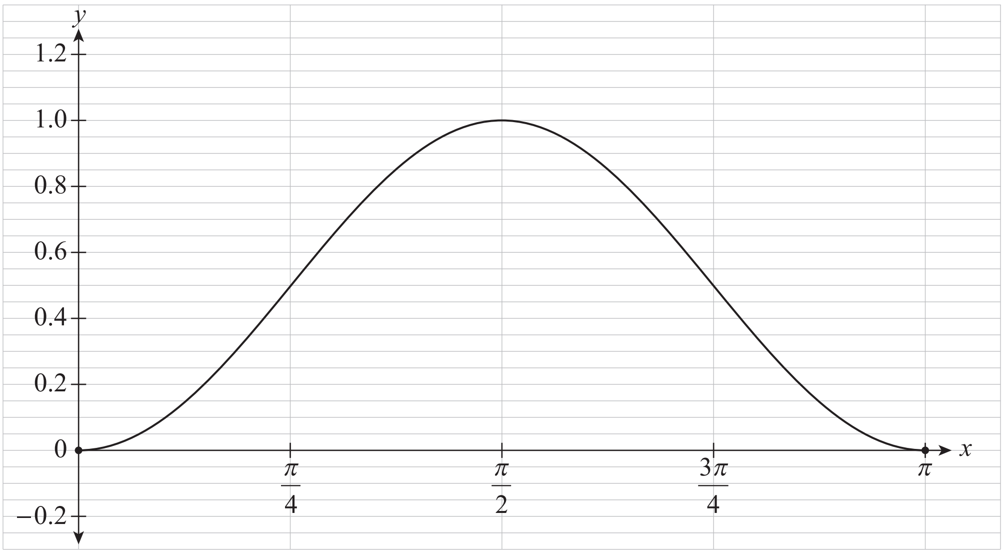
The probability density function $f(t) = m \times t^2 (n-t)^2$, where $0 \le t \le n$ and $m,\ n > 0$, can be used to model the probability that *any* type of mobile phone will have been replaced at time t years after it was purchased.

(c) Find the value of m in terms of n.

(4 marks)

Question 16 (12 marks)

(a) Below is a graph of $y = \sin^2 x$, for $0 \le x \le \pi$.



(i) Given that $1 - \cos^2 x = \sin^2 x$, show that $\frac{\mathrm{d}}{\mathrm{d}x}(x - \sin x \cos x) = 2\sin^2 x$.

(2 marks)

(ii) Hence show that the *exact* value of the area between the x-axis and the graph of $y = \sin^2 x$ is equal to $\frac{\pi}{2}$ on the interval $0 \le x \le \pi$.

(3 marks)

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(b) The ‘average value’ of any function $f(x)$ on the interval $a \le x \le b$ can be calculated, using the following formula:

$$\text{average value} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

Using the information provided in part (a)(ii), find the average value of $f(x) = \sin^2 x$ on the interval $0 \le x \le \pi$.

(1 mark)

(c) The ‘root mean square’ of a function is calculated by finding the average value of *the square of* the function over an interval, and then taking the square root of that average.

The root mean square of any function $g(x)$ on the interval $a \le x \le b$ is given by the following formula:

$$\text{root mean square} = \sqrt{\frac{1}{b-a}\int_a^b \left[g(x)\right]^2 \mathrm{d}x}.$$

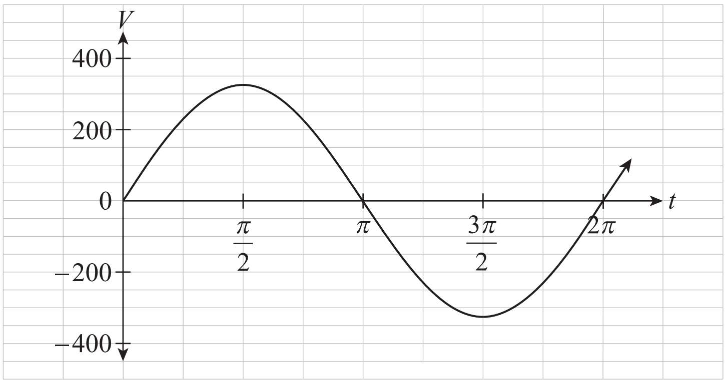
Show that the root mean square of $g(x) = \sin x$ is equal to $\frac{1}{\sqrt{2}}$ on the interval $0 \le x \le \pi$.

(1 mark)

In Australia, a model for the voltage sine function is

$$V(t) = 325\sin t,$$

where V is the voltage at time t, as shown below.



(d) Household electrical voltage is specified by the root mean square of the voltage sine function.

(i) Using the formula given in part (b), find the average value of $\left[V(t)\right]^2$ on the interval $0 \le t \le \pi$.

(3 marks)

(ii) Hence, using the formula given in part (c), find the household electrical voltage in Australia by calculating the root mean square of $V(t)$ on the interval $0 \le t \le \pi$.

(2 marks)