SACE BOARD
SOUTH AUSTRALIA

Stage 2 Mathematical Methods

Sample examination questions - 3

Government of South Australia

Question 1 (9 marks)

(a) For the functions below, determine $\frac{\mathrm{d}y}{\mathrm{d}x}$. You do not need to simplify your answers.

(i) $y = 7\cos(x+2)$.

(1 mark)

(ii) $y = e^{2x}(5x+6)^4$.

(3 marks)

(iii) $y = \frac{\sqrt{x^3+2}}{8\sin x}$.

(3 marks)

(b) Find $\int 8-\frac{1}{x}\,\mathrm{d}x$ for $x>0$.

(2 marks)

Question 2 (10 marks)

Analysis of the *Finance Update* magazine indicates that the use of the word 'exponential' in this magazine has been increasing since 1970. The number of times, $N(t)$, that the word 'exponential' is used in this magazine in a year, t years after 1970, can be modelled by the function below.

$$N(t) = 8.09e^{0.0812t}$$

(a) (i) Evaluate $N(10)$.

(1 mark)

(ii) Explain the meaning of $N(10)$ in the context of this model.

(1 mark)

(b) Calculate the number of times that the word 'exponential' was used in this magazine in 2018, according to this model.

(2 marks)

(c) (i) Find $N'(t)$.

(1 mark)

(ii) Evaluate $N'(51)$.

(1 mark)

(iii) Explain the meaning of $N'(51)$ in the context of this model.

(2 marks)

(d) (i) Describe what happens to $N(t)$ as t becomes very large.

(1 mark)

(ii) Suggest *one* reason why this model is ***not*** suitable for long-term predictions of the number of times that the word 'exponential' will be used in this magazine in a year.

(1 mark)

Question 3 (12 marks)

Phuong is comparing two types of chocolate bar: the Winks bar and the Monte bar.

(a) She first considers the **Winks** bar.

The Winks bar is sold in *packets* containing four randomly selected individual bars. The net weight, W grams, of a packet of Winks bars can be modelled by a normal distribution with mean $\mu_W = 580$ grams and standard deviation $\sigma_W = 10.2$ grams. Each packet is labelled as having a net weight of 550 grams.

(i) Determine the probability that the net weight of a randomly selected *packet* of Winks bars is:

(1) between 432 grams and 568 grams.

(1 mark)

(2) less than 550 grams.

(1 mark)

(3) exactly 575 grams.

(1 mark)

(ii) Phuong assumes that the net weights of *individual* Winks bars can be modelled by a normal distribution.

Does the information provided in part (a) support her assumption? Justify your answer.

(2 marks)

(b) Phuong now considers the **Monte** bar.

The net weights, M grams, of *individual* Monte bars can be modelled by a normal distribution with mean $\mu_M = 120$ grams and standard deviation σ_M grams.

(i) Find the value of k given $\Pr(Z > k) = 0.916$, where Z is normally distributed with mean $\mu_Z = 0$ and standard deviation $\sigma_Z = 1$.

(1 mark)

(ii) Hence show that $\sigma_M \approx 7.25$ grams given $\Pr(M > 110) = 0.916$.

(2 marks)

(iii) The Monte bar is sold in *packets* containing five randomly selected individual bars. Each packet is labelled as having a net weight of 550 grams.

Determine the probability that the net weight of a randomly selected packet of Monte bars is less than 550 grams.

(2 marks)

(c) Phuong randomly selects one packet of Winks bars and one packet of Monte bars.

Which *one* of these two packets is more likely to weigh less than its labelled net weight of 550 grams? Justify your answer.

(2 marks)

Question 4 (10 marks)

The continuous random variable X has the probability density function $f(x)$ shown below.

$$f(x)=\begin{cases}4xe^{2x} & 0\le x\le 0.5\\ 0 & \text{elsewhere}\end{cases}$$

Figure 1 shows the graph of $y=f(x)$ for $0\le x\le 0.5$.

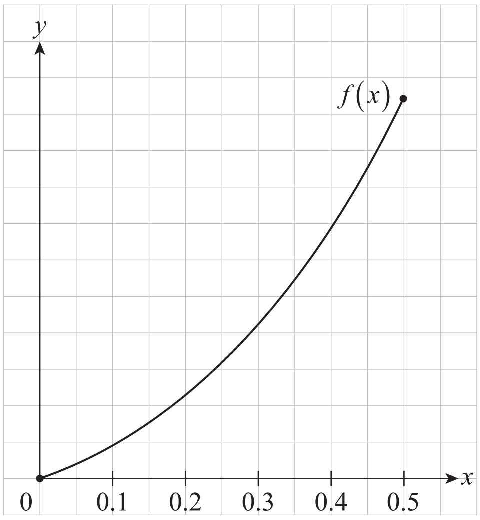

Figure 1

(a) The area between the graph of $y=f(x)$ and the x-axis over the domain $0\le x\le 0.5$ is 1 square unit.

State *one* other condition that $f(x)$ must satisfy, given that it is a probability density function.

(1 mark)

(b) Calculate $E(X)$.

(1 mark)

(c) To estimate $\Pr(0 \le X \le 0.3)$, an overestimate of the area between the graph of $y = f(x)$ and the x-axis from $x = 0$ to $x = 0.3$ is calculated, using three rectangles of equal width.

(i) *On the axes in Figure 1*, draw the rectangles used to obtain this overestimate. (1 mark)

(ii) Calculate this overestimate.

(2 marks)

(d) (i) Given $g(x) = e^{2x}(2x-1)$, show that $g'(x) = 4xe^{2x}$.

(2 marks)

(ii) Hence calculate the *exact* value of $\Pr(0 \le X \le 0.3)$.

(3 marks)

Question 5 (13 marks)

Consider the function $g(x) = 3\ln(3x) + \ln\left(\frac{1}{81x^4}\right)$.

(a) Express $g(x)$ in the form $a\ln(3x)$, where a is an integer.

(3 marks)

(b) Figure 2 shows the graph of $y = f(x)$ for $f(x) = \ln(3x)$.

On the axes in Figure 2, sketch the graph of $y = g(x)$.

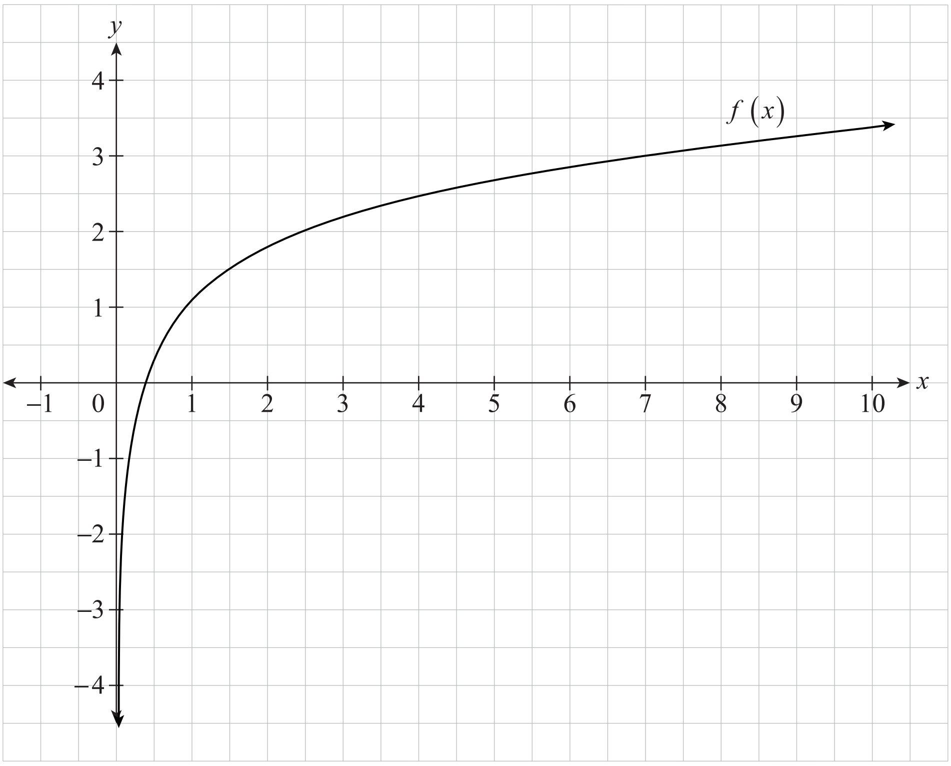

Figure 2

(2 marks)

(c) Comment on the relationship between the graphs of $y = f(x)$ and $y = g(x)$.

(1 mark)

Given $f(x) = \ln(3x)$, Figure 3 below shows the graph of $y = f(x)$, and its normal at $x = 2$. The point labelled A lies on this normal and has an x-coordinate of 1.

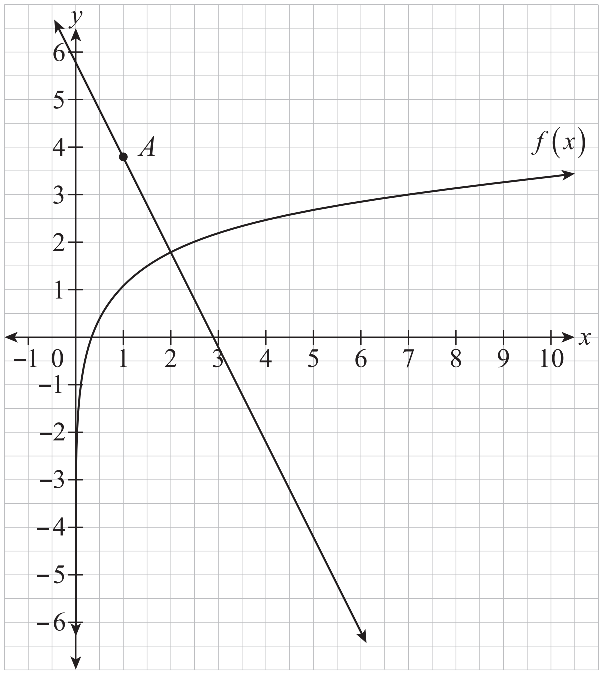

Figure 3

(d) (i) Find $f'(x)$.

(1 mark)

(ii) Hence show that the *exact* equation of the normal to the graph of the function $y = f(x)$ at $x = 2$ is $2x + y = 4 + \ln 6$.

(2 marks)

(e) Determine the *exact* y-coordinate of A.

(1 mark)

(f) Let the graph of $y = h(x)$ be the vertical translation, up k units, of the graph of $y = f(x)$, passing through A.

Determine the *exact* value of k in the form $p + \ln q$, where p and q are integers.

(3 marks)

Question 6 (9 marks)

A local library has historical data showing that 14% of all books borrowed from the library are returned late (i.e. after the due date).

(a) A random variable, X, represents whether a borrowed book is returned late ($X = 1$) or returned by the due date ($X = 0$).

State the type of distribution of X.

(1 mark)

The library staff took random samples of 10 borrowed books, and recorded the number of these books that were returned late. The number of books in a sample that were returned late, Y, can be modelled using a binomial distribution.

(b) Determine the probability that, in any one random sample of 10 borrowed books:

(i) none of the books was returned late.

(1 mark)

(ii) less than five of the books were returned late.

(2 marks)

The library staff decided to charge a fee when people return books late, in order to reduce the proportion of borrowed books that are returned late.

After 6 months, the library staff selected a random sample of 200 books that had been borrowed following the introduction of the fee, and recorded whether or not these books were returned late.

They found that 24 of these 200 books were returned late.

The library staff then calculated a 90% confidence interval for p, the true proportion of books that were returned late following the introduction of the fee. This confidence interval is:

$$0.0822 \le p \le 0.158.$$

(c) Explain why the library staff cannot claim with a 90% level of confidence that the introduction of the fee has reduced the proportion of books that are returned late.

(1 mark)

(d) Assume that the 90% confidence interval given is correctly calculated and is based on a random sample.

State *one* reason why the population proportion of books that are returned late may ***not*** be contained within this 90% confidence interval.

(1 mark)

The library staff would like to use their sample to make the claim, with a 'k% level of confidence', that the introduction of the fee has reduced the proportion of books that are returned late.

(e) What is the maximum value of k for which the library can make this claim? Give your answer correct to two significant figures.

(3 marks)

Question 7 (9 marks)

Figure 4 shows a graph of $y=f(x)$, where $f(x)=2\ln(2+\sin x)$ for $0\le x\le\frac{3\pi}{2}$.

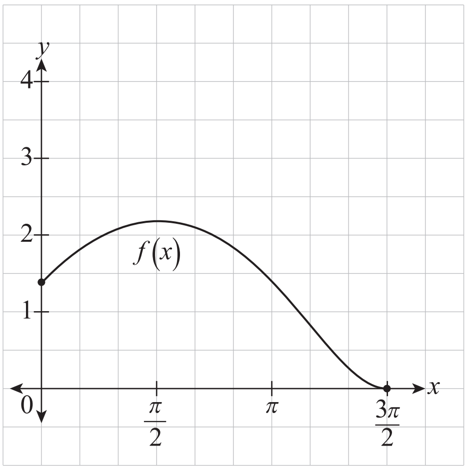

Figure 4

(a) Find $f'(x)$.

(1 mark)

(b) Show that $f''(x)=\dfrac{-2-4\sin x}{(2+\sin x)^2}$.

Note: $\sin^2 x+\cos^2 x=1$.

(3 marks)

(c) Hence, using an algebraic approach, find the *exact* coordinates of the inflection point for the graph of $y = f(x)$.

(3 marks)

(d) State the interval for which the shape of the graph of $y = f(x)$ is concave downwards.

(1 mark)

(e) *On the axes in Figure 4*, sketch the tangent to the graph of $y = f(x)$ that has the most negative slope.

(1 mark)

Question 8 (11 marks)

(a) Consider the discrete probability distribution shown in the table below.

x_i	1	2
$P(X = x_i)$	a	a

(i) State the value of a.

(1 mark)

(ii) Hence show that the mean of this distribution is $\frac{3}{2}$.

(1 mark)

(iii) Hence show that the standard deviation of this distribution is $\frac{1}{2}$.

(2 marks)

(b) The distribution in part (a) belongs to the family of discrete probability distributions defined in the table below, where n is a positive integer.

x_i	1	2
$P(X = x_i)$	a	na

(i) Show that $a = \frac{1}{n+1}$.

(1 mark)

(ii) Hence, find an expression for the mean of the distribution, in terms of n.

(1 mark)

The standard deviations of the distributions for four values of n are shown in the table below.

n	1	2	3	4
σ	$\frac{1}{2}$	$\frac{\sqrt{2}}{3}$	$\frac{\sqrt{3}}{4}$	$\frac{2}{5}$

(iii) Using the information in the table above, make a conjecture about the value of the standard deviation for any value of n, where n is a positive integer.

(1 mark)

(iv) Prove or disprove your conjecture.

(4 marks)

Question 9 (17 marks)

In one video-game tournament, teams are rated according to their relative skill level, where a higher rating indicates a higher skill level.

Let any team, A, have a rating of R_{A} points and any opposing team, B, have a rating of R_{B} points.

These ratings allow a simple model to be constructed using a normal distribution to approximate W, the probability that team A wins a match against team B.

Let X be a normally distributed variable with mean $\mu = 0$ and standard deviation $\sigma = 300$, where X can be used to calculate the probability that any team, A, wins a match against any team, B, when using the following model:

$$W = \Pr(\text{A wins a match against B}) = \Pr(X \le R_{\mathrm{A}} - R_{\mathrm{B}}).$$

The distribution of X is shown in Figure 5.

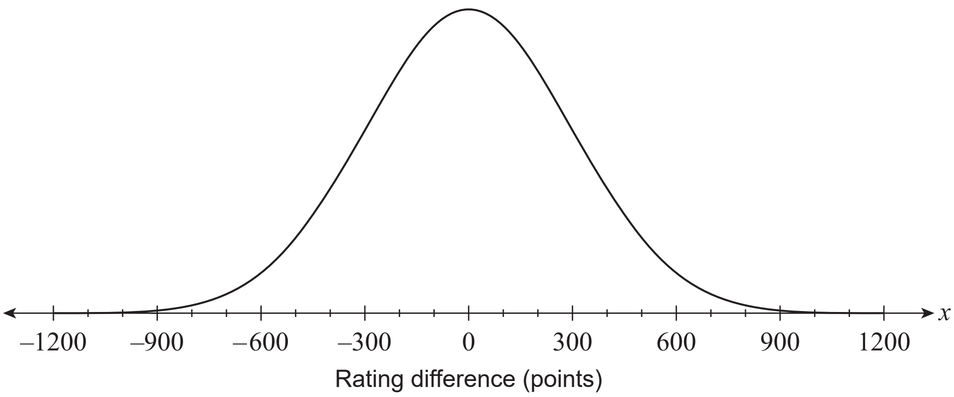

Figure 5

(a) (i) For $R_{\mathrm{A}} = 1500$ points and $R_{\mathrm{B}} = 1000$ points, find the probability that team A wins a match against team B.

(2 marks)

(ii) *On the distribution in Figure 5*, represent this probability by shading the appropriate area.

(1 mark)

Team ratings can change, depending on the outcome of a match. If any team, A, wins a match, its new rating, R_{new}, will be calculated by the following model:

$$R_{\text{new}} = R_{\mathrm{A}} + 32(1 - W).$$

(b) What is the limit on the number of points by which the rating of any team, A, can increase as a result of winning a single match? Explain your answer.

(2 marks)

(c) (i) For $R_{\mathrm{A}} = 1600$ points and $R_{\mathrm{B}} = 1900$ points, find the probability that team A wins a match against team B.

(1 mark)

(ii) For $R_{\mathrm{A}} = 1600$ points and $R_{\mathrm{B}} = 1900$ points, calculate what team A's new rating will become if it wins a match against team B.

(2 marks)

(d) (i) For $R_{\text{A}} = 980$ points and $R_{\text{B}} = 425$ points, calculate what team A's new rating will become if it wins a match against team B.

(3 marks)

(ii) Hence, do you think that this model unreasonably increases the rating of a team when it wins a match against a team that has a much lower rating? Explain your answer.

(2 marks)

(e) Suppose that team A has a rating of 1600 points and wants to increase its rating by at least 20 points through winning a single match.

Determine the minimum rating of the opposing team, team B, that team A must win the match against in order to achieve this.

(4 marks)