South Australian
Certificate of Education

1

Specialist Mathematics
2022

Question booklet 1

Questions 1 to 7 (55 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on page 15 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials
- Question booklet 1
- Question booklet 2
- Formula sheet
- SACE registration number label

Instructions
- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams

Total time: 130 minutes
Total marks: 100

Attach your SACE registration number label here	**Graphics calculator** 1. Brand ____________ Model ____________ 2. Brand ____________ Model ____________

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Question 1 (6 marks)

(a) Use integration by parts to find $\int x \sin x \, dx$.

(3 marks)

Consider the image of a pearl shown in Figure 1.

This image has been removed for copyright reasons.

Figure 2 shows the graph of $y = \sqrt{x \sin x}$, for $0 \le x \le \pi$, which models the top half of the cross-section of this pearl, outlined in Figure 1.

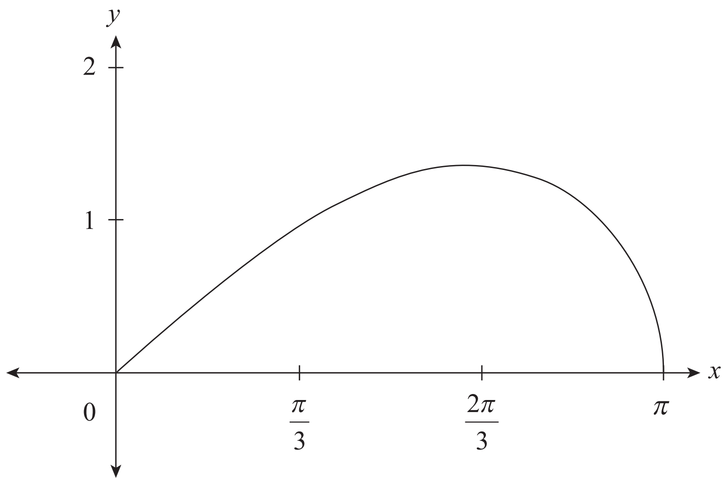


Figure 2

(b) The shape of the pearl can be obtained by rotating the curve in Figure 2 about the x-axis for $0 \leq x \leq \pi$.

Show that, according to the model, the exact volume of the pearl is π^2 cubic units.

(3 marks)

Question 2 (6 marks)

(a) (i) Write each of the following complex numbers in polar form.

(1) $z = -\sqrt{2} + \sqrt{2}\,i$

(1 mark)

(2) $w = \sqrt{6} - \sqrt{2}\,i$

(1 mark)

(ii) Hence find zw in polar form.

(1 mark)

(b) (i) Use de Moivre's theorem to write $(zw)^n$ in polar form, where n is a positive integer.

(1 mark)

(ii) Find the smallest positive value of n for which $(zw)^n$ is real and positive.

(2 marks)

Question 3 (7 marks)

Let $P(x) = x^n + 5x^2 + cx - 1$, where n is a positive integer and c is a real constant.

(a) If $(x+1)$ is a factor of $P(x)$, show that c is equal to either 3 or 5.

(3 marks)

(b) When $P(x)$ is divided by $(x-2)$, the remainder is 57.

Show that there is only one possible value of n.

(3 marks)

(c) Hence state the polynomial $P(x)$.

(1 mark)

Question 4 (10 marks)

Consider the curve defined by the parametric equations below.

$$\begin{cases} x(t) = 0.1e^t \cos t \\ y(t) = \sin t \end{cases} \quad \text{for } 0 \le t \le \frac{3\pi}{2}.$$

(a) Sketch the curve on the axes in Figure 3.

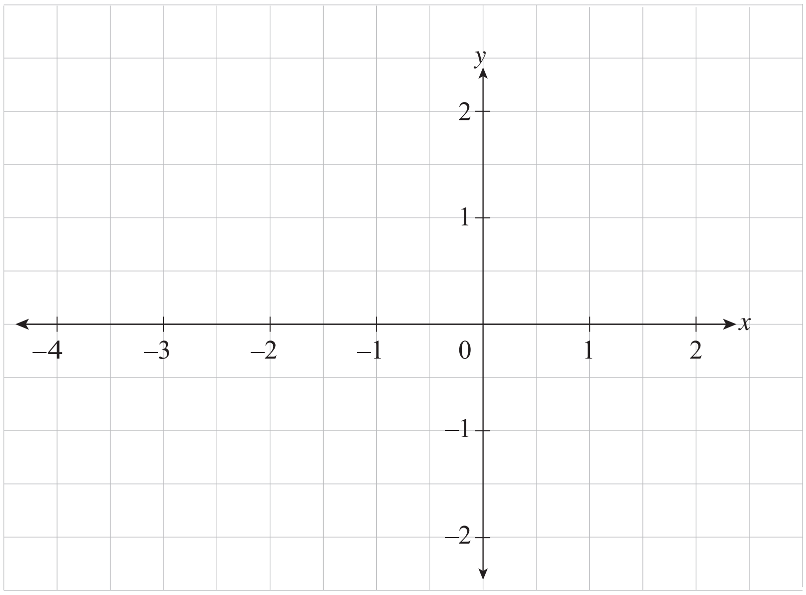


Figure 3 (3 marks)

(b) Consider a particle travelling along the curve you sketched on Figure 3. Its position at time t seconds is given by

$$\big(x(t), y(t)\big), \text{ where } 0 \le t \le \frac{3\pi}{2}.$$

(i) Show that the velocity of the particle at t seconds is given by

$$\boldsymbol{v} = \left[0.1e^t(\cos t - \sin t), \cos t\right].$$

(2 marks)

(ii) Show that the length of the path travelled by the particle is given by

$$\int_0^{\frac{3\pi}{2}} \sqrt{0.01e^{2t}(1-\sin 2t)+\cos^2 t}\,\mathrm{d}t.$$

(3 marks)

(iii) Hence calculate the length of the path travelled by the particle, correct to three significant figures.

(2 marks)

Question 5 (7 marks)

(a) Use mathematical induction to prove that for any positive integer n

$$1+\frac{1}{x}+\frac{1}{x^2}+\ldots+\frac{1}{x^n}=\frac{x^{n+1}-1}{x^n\left(x-1\right)}, \quad \text{if } x\neq 0,\ x\neq 1.$$

(5 marks)

(b) Hence show that, for any positive integer n:

$$1+\frac{1}{11}+\frac{1}{11^2}+...+\frac{1}{11^n}<1.1.$$

(2 marks)

Question 6 (9 marks)

(a) Figure 4 shows the triangle ABC, where $\overrightarrow{CA} = \boldsymbol{a}$ and $\overrightarrow{CB} = \boldsymbol{b}$.

Points P and Q lie on CA and AB respectively, such that

$$CP = kPA$$
$$BQ = kQA$$

where k is a positive constant.

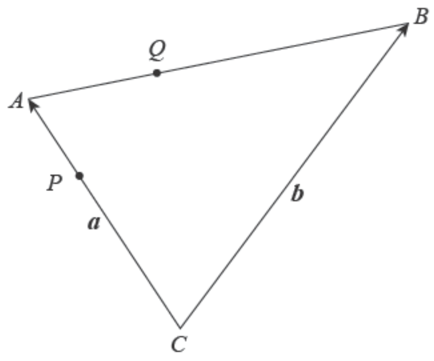


Figure 4

Write the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(i) $\overrightarrow{AB}$

(1 mark)

(ii) $\overrightarrow{PA}$

(1 mark)

(b) (i) Using vectors, show that PQ is parallel to CB.

(2 marks)

(ii) Using vectors, show that the area of triangle APQ is $\dfrac{1}{2(k+1)^2}|\boldsymbol{a} \times \boldsymbol{b}|$.

(2 marks)

(c) If $|\boldsymbol{a} \times \boldsymbol{b}| = 9$ and $k = 4$:

(i) find the area of triangle APQ

(1 mark)

(ii) find the area of quadrilateral $PQBC$.

(2 marks)

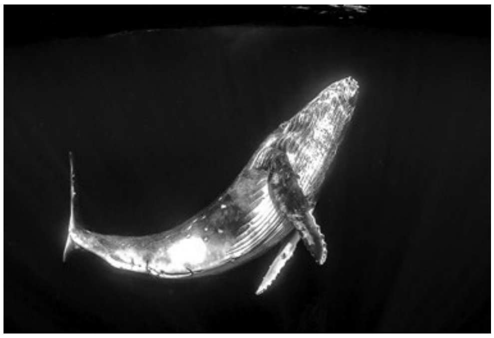
Question 7 (10 marks)

A whale that is initially 500 metres below the surface of the ocean rises towards the surface.

At time t minutes, the whale is at a depth of y kilometres below the surface.

The rate of change of depth of the whale is given by the differential equation below.

$$\left(1+t^2\right)\frac{dy}{dt}=2y^2t$$



Source: © John Natoli | iStockphoto.com

Figure 5 shows the slope field for the solutions to this differential equation.

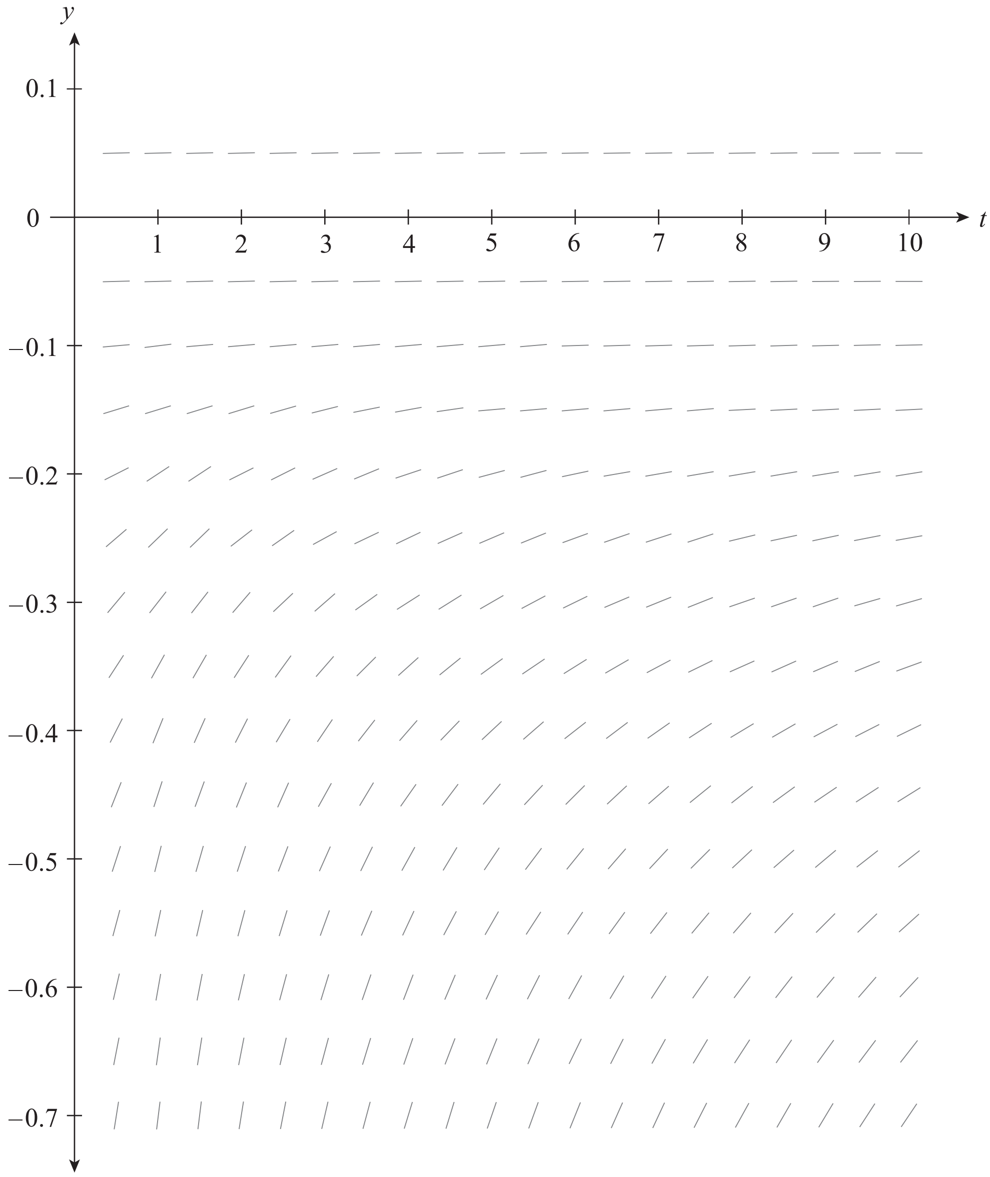


Figure 5

(a) On Figure 5, draw the solution curve for the differential equation using the initial condition $t = 0$ and $y = -0.5$. (3 marks)

(b) Use integration techniques to show that the solution to the differential equation when $y(0) = -0.5$ is

$$y = -\frac{1}{\ln\left(1+t^2\right)+2}.$$

(4 marks)

Question 7 continues on page 14.

Before reaching the surface of the ocean, the whale dives. The path that the whale now follows is given by:

$$y = -\frac{1}{1 + e^{-\frac{1}{2}(t-10)}}$$

where the depth below the surface of the ocean is y kilometres and t is time measured in minutes.

(c) If the whale starts its dive at $t = 0$:

(i) find the approximate depth to which the whale dives

(1 mark)

(ii) find the time at which the whale is diving at the greatest rate

(1 mark)

(iii) find the depth, y, of the whale at the time found in part (c)(ii).

(1 mark)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 4(b)(ii) continued).

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Specialist Mathematics
2022

Question booklet 2

Questions 8 to 10 (45 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on page 11 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

Copy the information from your SACE label here

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Question 8 (15 marks)

Consider the planes P_1 and P_2 that are defined by the equations below.

$$P_1: \ 2x + y - z = 1$$
$$P_2: \ 2x + 3y - z = 7$$

(a) (i) Clearly stating all row operations, show that P_1 and P_2 intersect at l_1, which has the following parametric equations:

$$\begin{cases} x = t \\ y = 3 \\ z = 2 + 2t \end{cases} \quad \text{where } t \text{ is a real parameter.}$$

(3 marks)

(ii) Show that the points $A\ (0, 3, 2)$ and $B\ (4, 3, 10)$ are on l_1.

(1 mark)

(iii) The plane P_3 is defined by the following equation: $4x + 3y - 2z = 63$.

Show that l_1 is parallel to P_3.

(2 marks)

(b) From part (a)(iii), the equation for P_3 is: $4x + 3y - 2z = 63$.
Point $Q\ (10, 9, 2)$ is on P_3.

(i) The line l_2 is normal to P_3 through Q.
Find the equation of l_2.

(2 marks)

(ii) Show that l_2 meets l_1 at C, where C is the midpoint of AB.

(3 marks)

(iii) Find the distance from l_1 to P_3.

(2 marks)

The line l_2 meets the plane $P_4: 4x + 3y - 2z = -63$ at T, as shown in Figure 6.

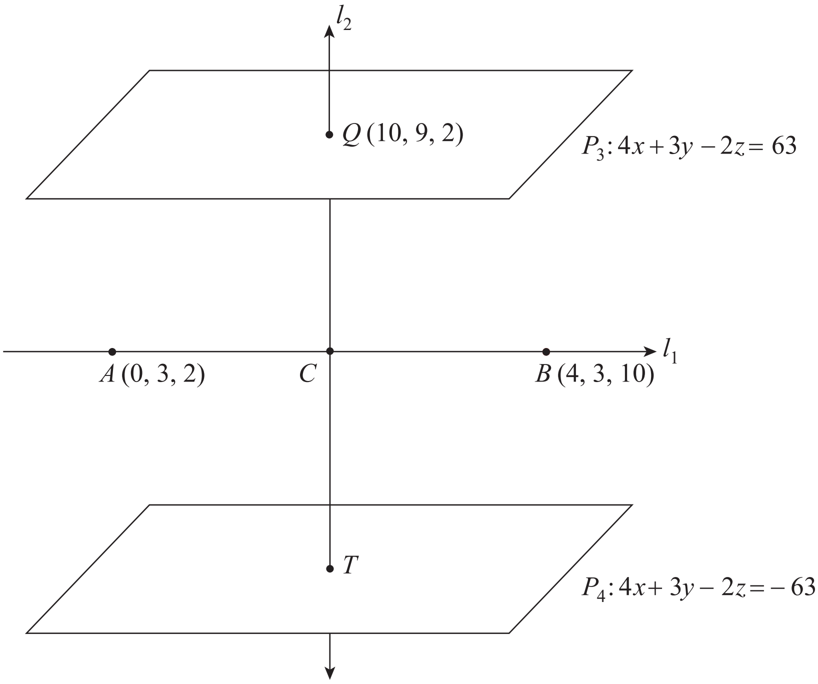


Figure 6

(c) Tick the appropriate box to complete the following statement:

The area of triangle ABT is

☐ less than the area of triangle ABQ.

☐ the same as the area of triangle ABQ.

☐ greater than the area of triangle ABQ.

Justify your answer.

(2 marks)

Question 9 (15 marks)

Consider $f(x) = \dfrac{x^2 - 1}{x + 2}$.

(a) Show that $f(x) = x - 2 + \dfrac{3}{x+2}$.

(1 mark)

(b) Sketch the graph of $y = f(x)$ on Figure 7 below.

Clearly label all asymptotes and the axes intercepts.

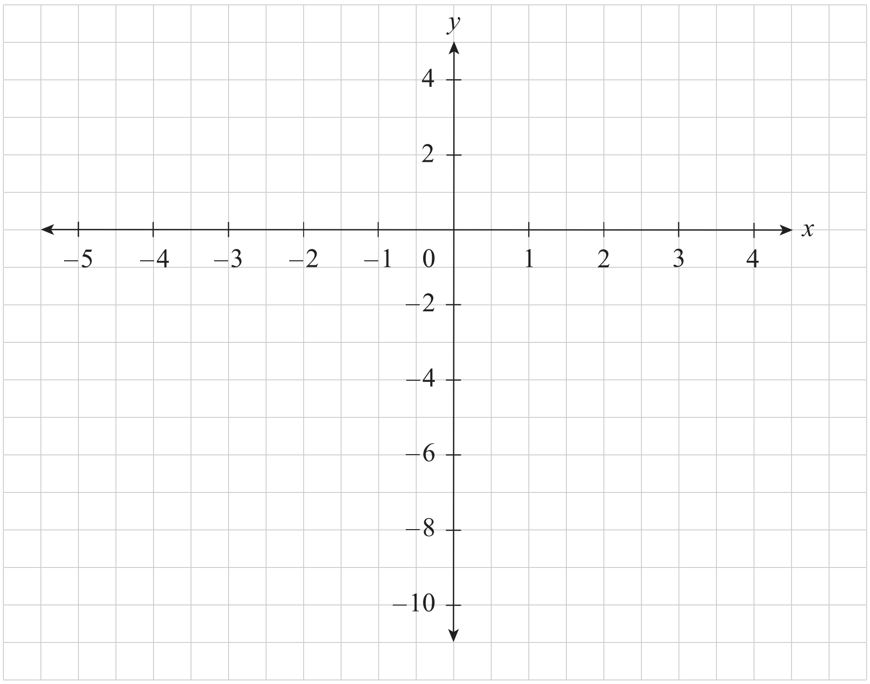


Figure 7 (4 marks)

(c) (i) On Figure 7 above, sketch and clearly label the graph of $y = f(|x|)$. (2 marks)

(ii) State the interval for which $f(|x|) > f(x)$ for $x > -2$.

(1 mark)

(d) (i) Show that the expression for finding the area between $f(|x|)$ and $f(x)$ for $x > -2$ is given by

$$\int_{-1}^{0} -2x + \frac{6x}{4-x^2}\,dx.$$

Note that $|x| = -x$ for $x \leq 0$.

(4 marks)

(ii) Hence show that the exact value of the area between $f(|x|)$ and $f(x)$ is

$$1 + 3\ln\left(\frac{3}{4}\right).$$

(3 marks)

Question 10 (15 marks)

(a) (i) On the Argand diagram in Figure 8:

- mark and label a point to represent a complex number z
- hence mark and label the point representing $z+2$.

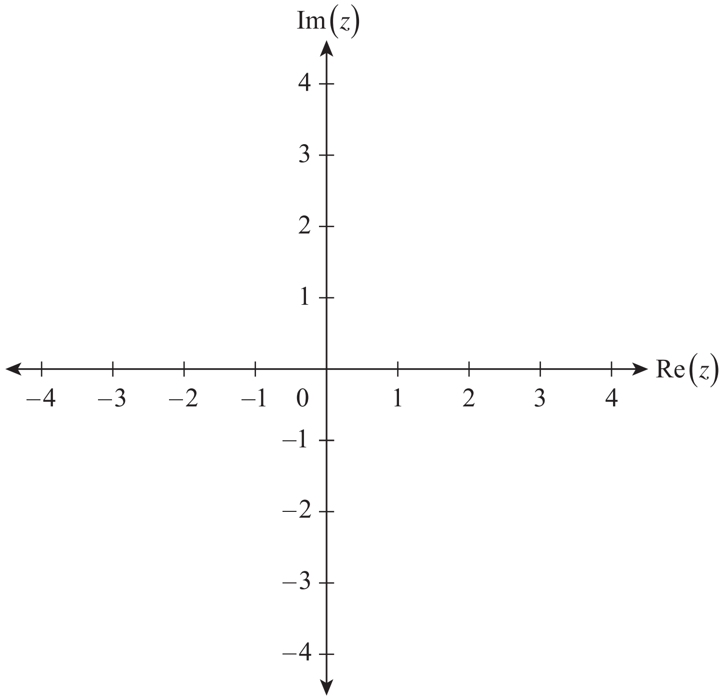


Figure 8 (1 mark)

(ii) Explain why $|z|+|z+2|\geq 2$.

(2 marks)

(iii) Explain why the only solutions for $|z|+|z+2|=2$ are real.

(1 mark)

(b) (i) On the Argand diagram in Figure 9:

(1) draw the set of all complex numbers z such that $|z+2|=|z|$. (2 marks)

(2) mark a point P, representing a complex number z such that $|z+2|=|z|$ and $\text{Im}(z)>0$. (1 mark)

(3) mark the point Q, representing $z+2$. (1 mark)

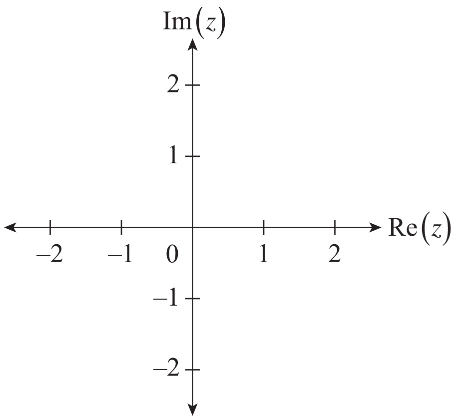


Figure 9

(ii) Let $\angle POQ=\theta$.

Show that $\dfrac{z}{z+2}=\text{cis}\,\theta$.

(2 marks)

Question 10 continues on page 10.

(c) (i) Using z from part (b), and $\theta = \frac{\pi}{6}$, write $\frac{z}{z+2}$ in Cartesian form.

(1 mark)

(ii) Hence show that $z = -1 + \left(2 + \sqrt{3}\right)i$.

(2 marks)

(iii) Using Figure 9 or otherwise, show that z may be written in polar form as $z = \operatorname{cosec}\frac{\pi}{12}\operatorname{cis}\frac{7\pi}{12}$.

(2 marks)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (eg. 8(b)(iii) continued).

SPECIALIST MATHEMATICS FORMULA SHEET

Circular functions

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$

$1 + \cot^2 A = \operatorname{cosec}^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$\sin 2A = 2 \sin A \cos A$

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

$\sin A \pm \sin B = 2 \sin \frac{1}{2}(A \pm B) \cos \frac{1}{2}(A \mp B)$

$\cos A + \cos B = 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$

$\cos A - \cos B = -2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)$

Matrices and determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

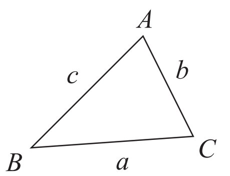
$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Measurement

Area of sector, $A = \frac{1}{2}r^2\theta$, where θ is in radians.

Arc length, $l = r\theta$, where θ is in radians.

In any triangle ABC:



Area of triangle $= \frac{1}{2}ab \sin C$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$a^2 = b^2 + c^2 - 2bc \cos A$

Quadratic equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Distance from a point to a plane

The distance from (x_1, y_1, z_1) to $Ax + By + Cz + D = 0$ is given by

$$\frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Derivatives

$f(x) = y$	$f'(x) = \dfrac{dy}{dx}$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$

Properties of derivatives

$$\frac{d}{dx}\big(f(x)\,g(x)\big) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\frac{d}{dx}f(g(x)) = f'(g(x))\,g'(x)$$

Arc length along a parametric curve

$l = \int_a^b \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}\, dt$, where $a \le t \le b$.

Integration by parts

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

Volumes of revolution

About x axis, $V = \int_a^b \pi y^2\,dx$, where y is a function of x.

About y axis, $V = \int_c^d \pi x^2\,dy$, where y is a one-to-one function of x.