1

Specialist Mathematics
2024

Question booklet 1

Questions 1 to 7 (55 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on page 16 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials
- Question booklet 1
- Question booklet 2
- Formula sheet
- SACE registration number label

Instructions
- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams

Total time: 130 minutes
Total marks: 100

The SACE Board of South Australia acknowledges that this examination was created on Kaurna Land. We acknowledge First Nations Elders, parents, families, and communities as the first educators of their children, and we recognise and value the cultures and strengths that First Nations students bring to the classroom. We respect the unique connection and relationship that First Nations peoples have to Country, and their ever-enduring cultural heritage.

Attach your SACE registration number label here

Graphics calculator

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Question 1 (6 marks)

Consider the polynomial $P(z)=z^4-5z^3+az^2+bz-20$, where a and b are real constants.

(a) Find a real quadratic factor of $P(z)$, given that $3+i$ is a zero.

(2 marks)

(b) (i) Given that $P(z)$ has a factor of $(z-1)$, show that $a+b=24$.

(1 mark)

(ii) When $P(z)$ is divided by $(z+1)$, there is a remainder of -34. Show that $a-b=-20$.

(1 mark)

(iii) State the values of a and b.

(1 mark)

(c) Write $P(z)$ as a product of a real quadratic and real linear factors.

(1 mark)

Question 2 (7 marks)

Consider the planes P_1 and P_2 in 3D space.

$$P_1 : x + 2y - 3z = 9$$
$$P_2 : x + y + 2z = 5$$

(a) Show that these planes are not parallel.

(1 mark)

(b) The augmented matrix for the planes in part (a) is $\begin{bmatrix} 1 & 2 & -3 & : & 9 \\ 1 & 1 & 2 & : & 5 \end{bmatrix}$.

Use a clearly defined row operation or row operations to show that these planes meet along a common line, l_1, with parametric equations,

$$\begin{cases} x = 1 - 7t \\ y = 4 + 5t \\ z = t \end{cases}, \text{ where } t \text{ is a real parameter.}$$

(2 marks)

(c) Show that the point $A(-13, 14, 2)$ lies on l_1.

(1 mark)

Figure 1 shows P_1, P_2, l_1, and the point $Q(-10,\ 12,\ 0)$.

Point Q does not lie on P_1 and does not lie on P_2 .

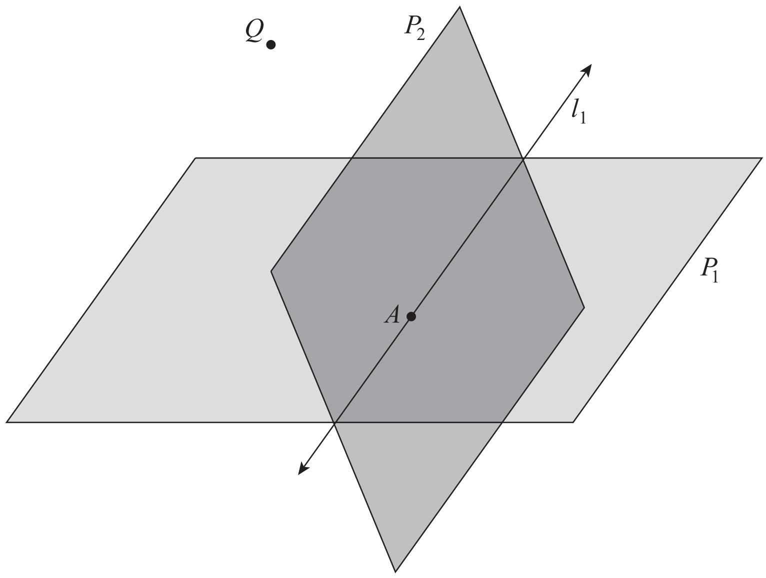

Figure 1

(d) Find the equation of plane P_3 that contains both l_1 and Q.

(3 marks)

Question 3 (8 marks)

(a) Show that $\begin{bmatrix} i & a \\ 0 & i \end{bmatrix}^3 = \begin{bmatrix} -i & -3a \\ 0 & -i \end{bmatrix}$, where a is a real constant, and $i^2 = -1$.

(2 marks)

(b) Prove by mathematical induction that $\begin{bmatrix} i & a \\ 0 & i \end{bmatrix}^n = \begin{bmatrix} i^n & nai^{n-1} \\ 0 & i^n \end{bmatrix}$ for all positive integers n.

(4 marks)

(c) Hence, find $\begin{bmatrix} i & a \\ 0 & i \end{bmatrix}^3 - \begin{bmatrix} i & a \\ 0 & i \end{bmatrix}^{2027}$.

(2 marks)

Question 4 (7 marks)

Figure 2 shows the triangle OAB. The point M is the midpoint of OA, and the point N is the midpoint of OB.

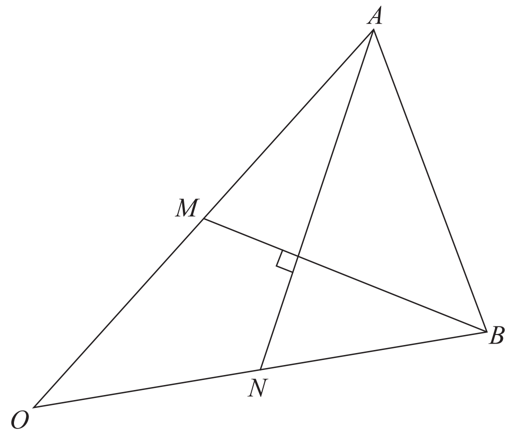

Figure 2

Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. The vectors $\overrightarrow{AN}$ and $\overrightarrow{BM}$ are perpendicular.

(a) (i) Show that $\overrightarrow{AN} = \frac{1}{2}\boldsymbol{b} - \boldsymbol{a}$.

(1 mark)

(ii) Find $\overrightarrow{BM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(1 mark)

(b) Use the expressions found in part (a) and the fact that $\overrightarrow{AN}$ and $\overrightarrow{BM}$ are perpendicular to show that $|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 = \frac{5}{2}\boldsymbol{a}\cdot\boldsymbol{b}$.

(2 marks)

(c) Using the cosine rule in triangle OAB, show that $\left|\overrightarrow{AB}\right|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\,\boldsymbol{a}\cdot\boldsymbol{b}$.

(2 marks)

(d) Use the results of parts (b) and (c) to show that $\left|\overrightarrow{AB}\right|^2 = \frac{1}{5}\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2\right)$.

(1 mark)

Question 5 (10 marks)

Consider the function $f(x) = \ln\left(\sqrt{x-2}+2\right)$ for $x \geq 2$.

(a) Draw the curve of $y = \ln\left(\sqrt{x-2}+2\right)$ on the axes in Figure 3.

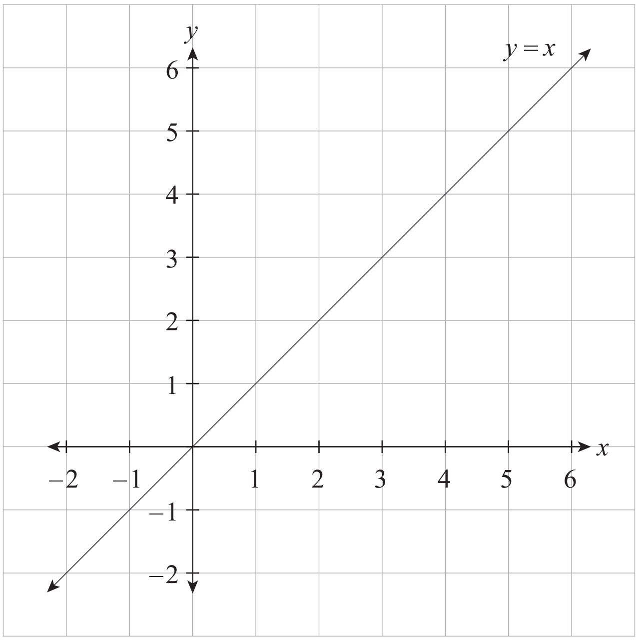

Figure 3 (2 marks)

(b) (i) Show that the inverse, $f^{-1}(x)$, of $y = \ln\left(\sqrt{x-2}+2\right)$ for $x \geq 2$ is $f^{-1}(x) = \left(e^{x}-2\right)^{2}+2$.

(2 marks)

(ii) State the exact domain of $f^{-1}(x)$.

(1 mark)

(iii) Draw $y = f^{-1}(x)$ on the axes in Figure 3, shown in part (a). (2 marks)

(c) The volume of the solid of revolution formed when the curve $y = f^{-1}(x)$ is rotated 2π about the y axis from $y = 2$ to $y = 5$ is $V = \pi \int_2^5 \left(\ln\left(\sqrt{y-2}+2\right)\right)^2 dy$.

(i) Using Figure 3 and (b)(iii), explain why the volume of the solid may also be given by

$$V = \pi \int_2^5 \left(\ln\left(\sqrt{x-2}+2\right)\right)^2 dx.$$

(1 mark)

(ii) Evaluate the volume of the solid, correct to 3 significant figures.

(2 marks)

Question 6 (8 marks)

The solutions of $z^6 = -8$ are shown on the Argand diagram in Figure 4.

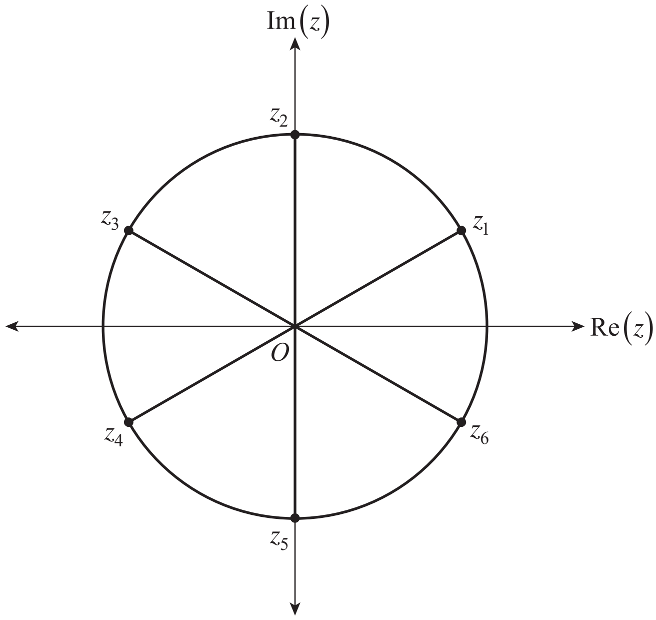

Figure 4

(a) (i) By using De Moivre's theorem or otherwise, write the solutions $z_1, z_2, z_3, z_4, z_5, z_6$ in $r cis\theta$ form below.

$z_1 =$ $z_2 =$ $z_3 =$

$z_4 =$ $z_5 =$ $z_6 =$

(2 marks)

(ii) Show that $\left|z_1 - z_6\right| = \sqrt{2}$.

(2 marks)

Figure 5 shows a triangle formed by joining O, z_1, and z_6, and a hexagon formed by joining $z_1, z_2, z_3, z_4, z_5, z_6$.

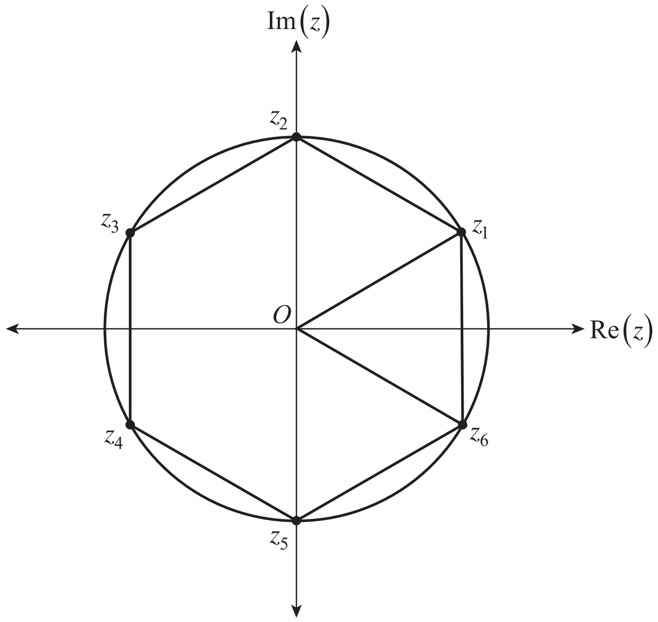

Figure 5

(b) (i) Show that the area of the triangle Oz_1z_6 is $\dfrac{\sqrt{3}}{2}$.

(2 marks)

(ii) Find the *exact* area of the hexagon.

(1 mark)

(c) The solutions of $z^6 = k$, where k is a real constant, form a hexagon of area $9\sqrt{3}$.

Find a value for k.

(1 mark)

Question 7 (9 marks)

Consider the function $f(x) = \sin x \ln(\cos x)$, where $\cos x > 0$.

(a) Using integration by parts, show that
$\int \sin x \ln(\cos x)\,dx = -\cos x \ln(\cos x) + \cos x + c$, where c is a real constant.

(2 marks)

(b) Hence, show that $\int_0^{\frac{\pi}{3}} \sin x \ln(\cos x)\,dx = \frac{1}{2}\ln 2 - \frac{1}{2}$.

(3 marks)

Figure 6 shows the graph of $f(x) = \sin x \ln(\cos x)$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

The point P is on the curve of $y = f(x)$, where $x = \frac{\pi}{3}$.

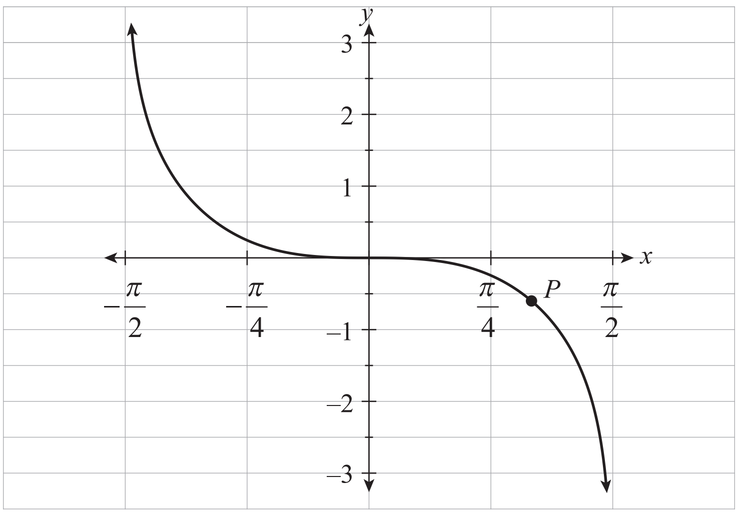

Figure 6

(c) On Figure 6, *clearly* draw and label the graph of $y = f(|x|)$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. (2 marks)

(d) Hence, find the *exact* area bounded by the curves $y = f(x)$ and $y = f(|x|)$, and the vertical lines $x = \frac{\pi}{3}$ and $x = -\frac{\pi}{3}$.

(2 marks)

 PLEASE TURN OVER

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 5(b)(i) continued).

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Question booklet 2

Questions 8 to 10 (45 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on pages 5, 9, and 12 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

2

Copy the information from your SACE label here

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Question 8 (16 marks)

A circle $(x-\pi)^2+(y-1)^2=1$ is drawn on Figure 7 below.

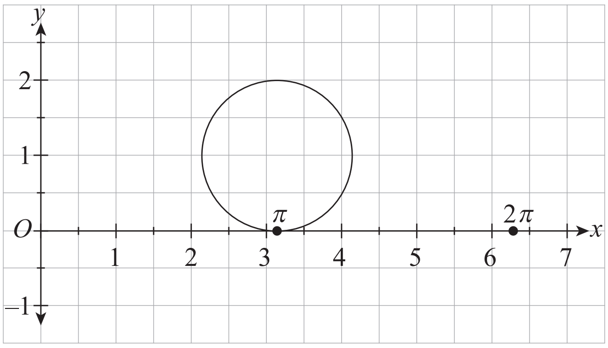


Figure 7

A parametric curve has the equations below.

$$\begin{cases} x(t)=t-\sin t \\ y(t)=1-\cos t \end{cases} \text{ for } 0\le t\le 2\pi$$

(a) Sketch the parametric curve on the axes in Figure 7 above. (3 marks)

(b) (i) Find $\frac{dy}{dx}$ for the circle.

(2 marks)

(ii) Find $\frac{dy}{dx}$ for the parametric curve drawn in part (a) in terms of t.

(2 marks)

(c) Hence, show that both curves have a common horizontal tangent at $P(\pi, 2)$.

(2 marks)

(d) From part (a), notice that the length of arc $OP >$ length of the line OP.

Consider the triangle OPA, where A is $(2\pi, 0)$.

Using the triangle inequality, show that the length of the parametric curve from 0 to 2π is greater than the circumference of the circle.

(3 marks)

Question 8 continues on page 4.

(e) (i) Show that the length of the parametric curve from $t = 0$ to $t = 2\pi$ is given by $\int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) dt$.

(3 marks)

(ii) Hence, find the length of the parametric curve.

(1 mark)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 8(e)(i) continued).

PLEASE TURN OVER

Question 9 (14 marks)

A small ball rolls down a path with velocity described by $\frac{dy}{dt} = -\frac{1}{10}e^{-0.5t}y^2$ metres per second, where $t \geq 0$ is measured in seconds and $y > 0$ is measured in metres.

(a) (i) Using integration techniques, show that the solution curve is $y = \frac{1}{c - 0.2e^{-0.5t}}$ where c is a constant.

(3 marks)

(ii) If the ball is initially at the position $t = 0$, $y = 10$, find the *exact* value y approaches as $t \to \infty$.

(2 marks)

(b) Draw the solution curve for an initial position of $t = 0$, $y = 10$ on the slope field shown in Figure 8.

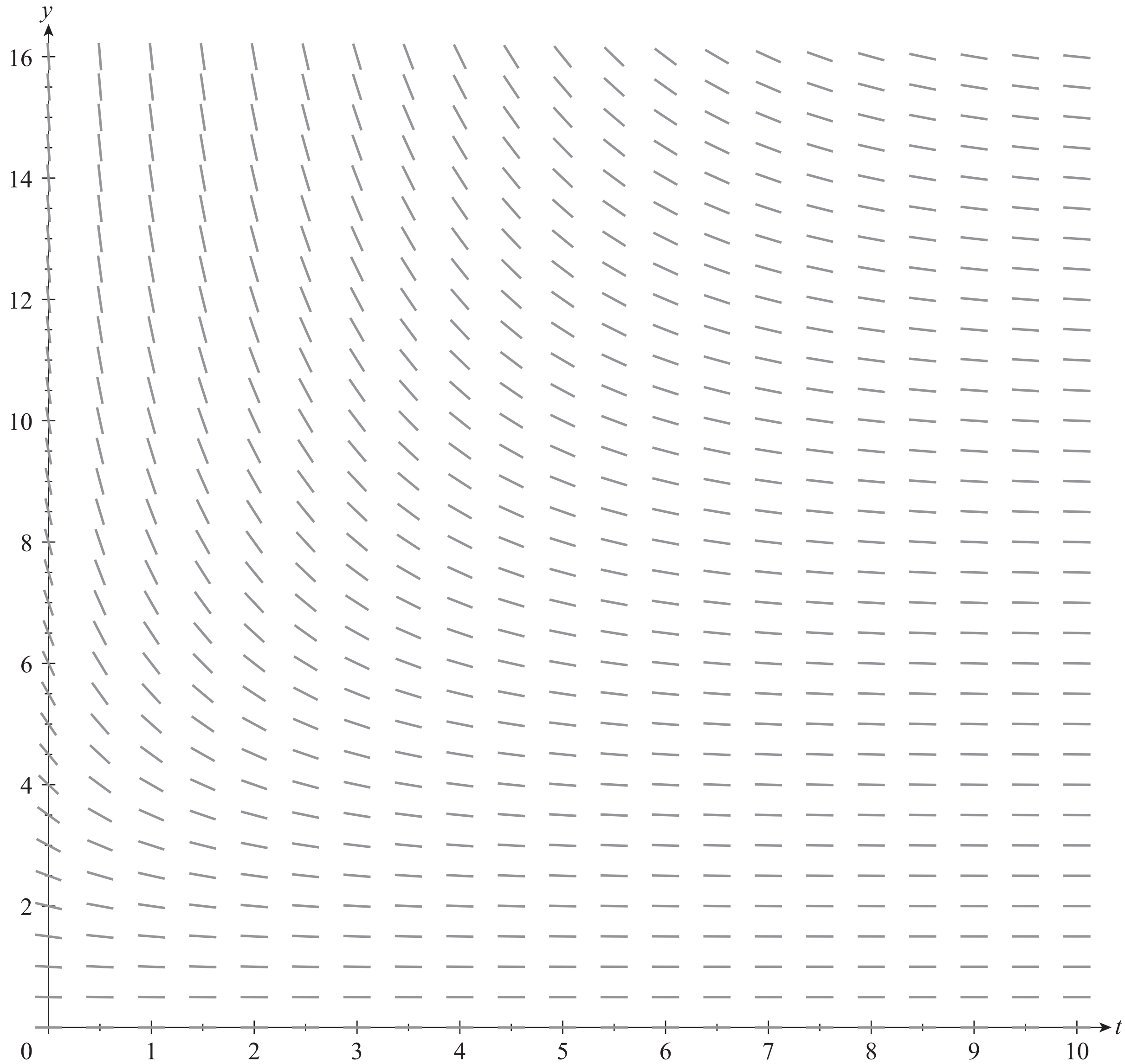


Figure 8 (3 marks)

Question 9 continues on page 8.

(c) Show that $\dfrac{d^2y}{dt^2} = -\dfrac{dy}{dt}\left(\dfrac{1}{2} + \dfrac{1}{5}e^{-0.5t}y\right)$.

(3 marks)

(d) Considering the curve from part (a)(ii) and part (b), find the *exact* value of t when $\dfrac{\left(\dfrac{d^2y}{dt^2}\right)}{\left(\dfrac{dy}{dt}\right)} = -1$.

(3 marks)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 9(c) continued).

Question 10 (15 marks)

The complex number $z_1 = 0.36\,cis\theta$, where $0 < \theta < \frac{\pi}{2}$, is shown on the Argand diagram in Figure 9.

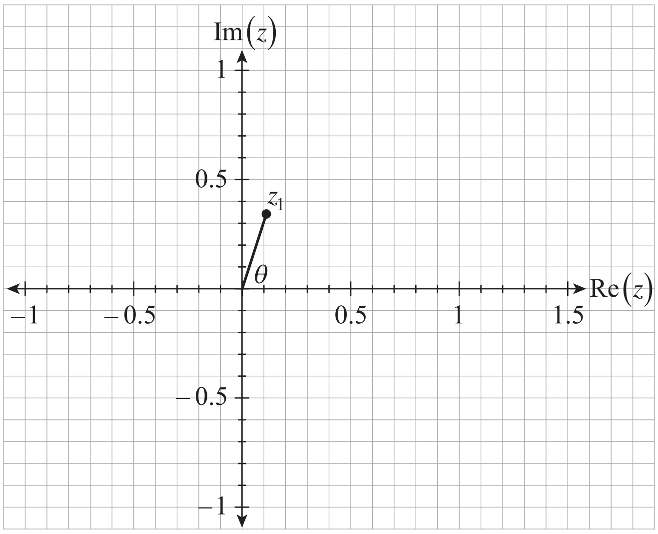


Figure 9

(a) (i) Find in polar form a complex number z_2 in the same quadrant such that $z_2^{\,2} = z_1$.

(2 marks)

(ii) Mark z_2 on Figure 9. (2 marks)

(b) (i) On Figure 9, mark the set S of complex numbers z such that $|z-1| \le 0.1$. (2 marks)

(ii) On Figure 9, mark the set T of complex numbers z such that $\text{Re}(z) \ge 0.9$. (2 marks)

(c) (i) Find the smallest positive integer n, such that $(0.36)^{\frac{1}{n}} \geq 0.9$.

(1 mark)

(ii) Hence, find in polar form a complex number w in the first quadrant such that $w^n = z_1$, where n is the integer found in part (c)(i) and $z_1 = 0.36 cis\theta$.

(2 marks)

(iii) Suppose $\theta = \frac{\pi}{5}$.

(1) Write w in cartesian form.

(1 mark)

(2) Is w in the set T? Justify your answer.

(1 mark)

(3) Is w within the set S? Justify your answer.

(2 marks)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 10(a)(i) continued).

SPECIALIST MATHEMATICS FORMULA SHEET

Circular functions

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$

$1 + \cot^2 A = \operatorname{cosec}^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2\sin A \cos A$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$\sin A \pm \sin B = 2\sin\frac{1}{2}(A \pm B)\cos\frac{1}{2}(A \mp B)$

$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

$\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$

Matrices and determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

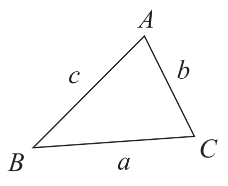
$A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Measurement

Area of sector, $A = \frac{1}{2}r^2\theta$, where θ is in radians.

Arc length, $l = r\theta$, where θ is in radians.

In any triangle ABC:



Area of triangle $= \frac{1}{2}ab\sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$a^2 = b^2 + c^2 - 2bc\cos A$

Quadratic equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Distance from a point to a plane

The distance from (x_1, y_1, z_1) to $Ax + By + Cz + D = 0$ is given by

$\dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Derivatives

$f(x) = y$	$f'(x) = \dfrac{dy}{dx}$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$

Properties of derivatives

$\dfrac{d}{dx}\big(f(x)\,g(x)\big) = f'(x)g(x) + f(x)g'(x)$

$\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$

$\dfrac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$

Arc length along a parametric curve

$l = \displaystyle\int_a^b \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}\,dt$, where $a \le t \le b$.

Integration by parts

$\displaystyle\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$

Volumes of revolution

About x axis, $V = \displaystyle\int_a^b \pi y^2\,dx$, where y is a function of x.

About y axis, $V = \displaystyle\int_c^d \pi x^2\,dy$, where y is a one-to-one function of x.