South Australian Certificate of Education

The purpose of this sample paper is to show the structure of the 130-minute Specialist Mathematics examination and the style of questions that might be used. The examination will consist of questions that assess a *selection* of the key questions and key concepts from across the six topics.

Specialist Mathematics
November 2020 sample paper

Question booklet 2

- Questions 8 to 10 (46 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on page 12 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

2

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SEQ	FIGURES	CHECK LETTER	BIN

Graphics calculator

1. Brand _____

Model _____

2. Brand _____

Model _____

Government of South Australia

Question 8 (16 marks)

Consider the system of equations shown below.

$$\begin{cases} x + 2y + 2z = 4 \\ 2x + y - 2z = 5 \\ 3x + 2y - 2z = 8 \end{cases}$$

(a) (i) Write this system as an augmented matrix.

(1 mark)

(ii) Stating all row operations, show that this system of equations has the following solutions:

$$\begin{aligned} x &= 2 + 2t \\ y &= 1 - 2t \\ z &= t \end{aligned}$$

where t is a real parameter.

(3 marks)

(b) Consider three planes in space, P_1, P_2 and P_3, defined by the system of equations below.

$$P_1: \quad x + 2y + 2z = 4$$
$$P_2: \quad 2x + y - 2z = 5$$
$$P_3: \quad 3x + 2y - 2z = 8$$

(i) Using the information given in part (a)(ii), show that the points $A(2, 1, 0)$ and $B(0, 3, -1)$ are common to all three planes.

(2 marks)

(ii) Show that P_1 and P_2 are perpendicular.

(2 marks)

(c) Figure 8 shows the point $C(0, 6, 2)$ on P_3, the point $D(12, -4, 0)$ on P_1, and the line l through C and D, intersecting P_2 at the point E.

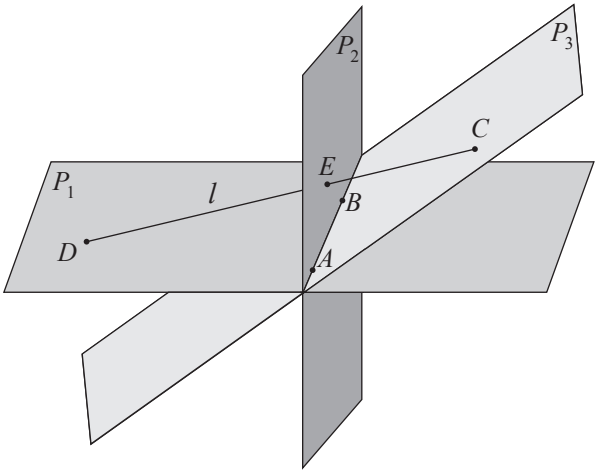


Figure 8

(i) Find, in parametric form, the equation of l.

(2 marks)

(ii) Find the coordinates of E.

(2 marks)

(iii) Find the distance from E to P_1.

(2 marks)

(d) The equations of P_1 and P_3 are used to model two hillsides that meet at a river, as shown in Figure 9.

$$P_1: \quad x + 2y + 2z = 4$$
$$P_3: 3x + 2y - 2z = 8$$

The river is modelled by the line where the two planes meet. A straight bridge, modelled by l, connects $C(0, 6, 2)$ to $D(12, -4, 0)$.

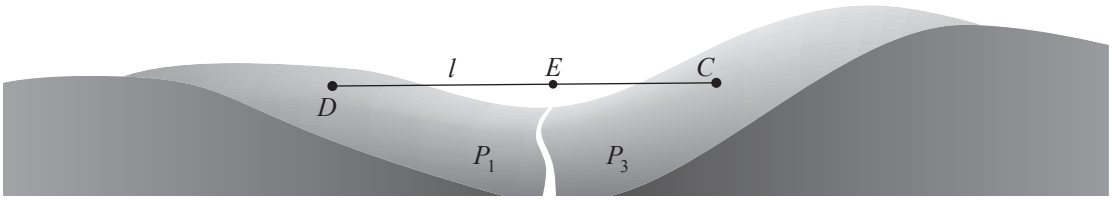


Figure 9

The point E, on the bridge, must be at least 1 unit from P_1 and at least 1 unit from P_3.

Does the model satisfy this condition? Show your calculations.

(2 marks)

Question 9 (15 marks)

(a) (i) Solve the equation $z^3 = 1$. Write the three solutions in polar form.

(2 marks)

(ii) Draw the solutions on the Argand diagram in Figure 10, labelling each solution in an anticlockwise direction α_1, α_2, and α_3, where α_1 is real.

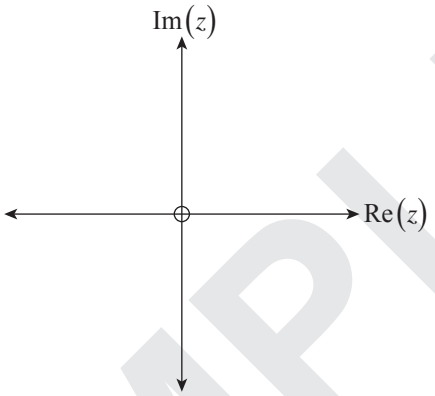


Figure 10

(2 marks)

(iii) Explain why α_2 lies on the line with equation $|z - \alpha_1| = |z - \alpha_3|$.

(1 mark)

(iv) Evaluate $\alpha_1 + \alpha_2 + \alpha_3$.

(1 mark)

(v) Evaluate $|\alpha_1|+|\alpha_2|+|\alpha_3|$.

(1 mark)

(vi) Find the *exact* value of $|\alpha_2-\alpha_3|$.

(1 mark)

(b) Consider the equation $(w-1)^3=1$.

(i) Show that the solutions to this equation are $w_1=1+\alpha_1$, $w_2=1+\alpha_2$, and $w_3=1+\alpha_3$, where α_1, α_2, and α_3 are the solutions to the equation $z^3=1$, as in part (a).

(1 mark)

(ii) Draw the solutions of $(w-1)^3=1$ on the Argand diagram in Figure 11.

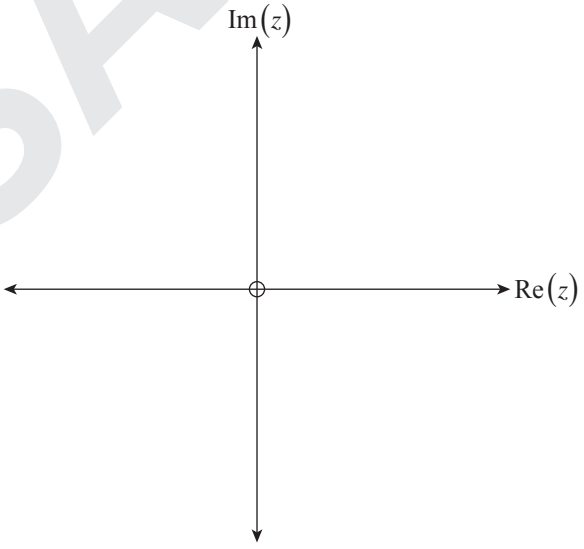


Figure 11

(2 marks)

(iii) Show that w_2 and w_3 lie on the line $\text{Re}(z) = \frac{1}{2}$.

(1 mark)

(iv) Evaluate $|w_1| + |w_2| + |w_3|$.

(1 mark)

(v) Find the *exact* value of $|w_1 - w_2| + |w_2 - w_3| + |w_3 - w_1|$.

(2 marks)

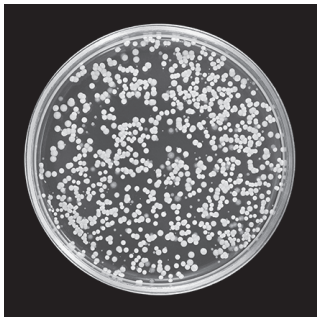
Question 10 (15 marks)

(a) In an experiment one type of bacterium, called alpha, was grown in a Petri dish.

The rate of change of the area in the Petri dish that was covered by alpha bacteria can be modelled by the differential equation

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}A\left(\frac{50 - A}{50}\right)$$

where A is area in cm^2 and t is time in days.



Source: © Satirus | Shutterstock.com

(i) At $t = 0$, the area in the Petri dish that was covered by alpha bacteria was 1 cm^2.

On the slope field in Figure 12, draw the solution curve.

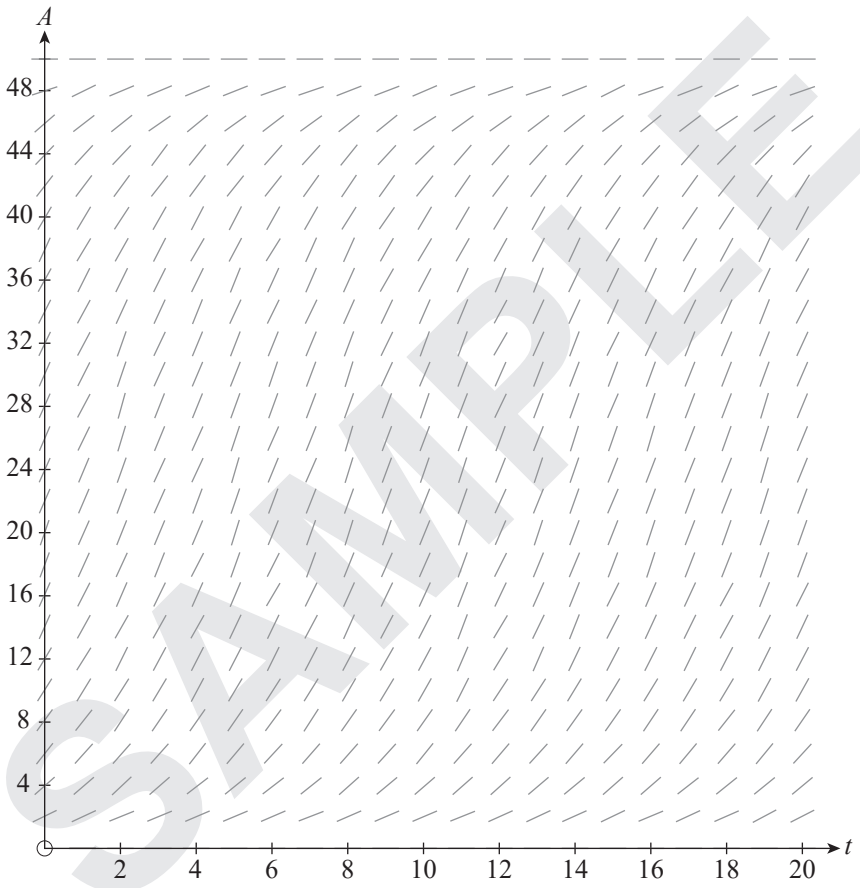


Figure 12

(3 marks)

(ii) Show that $\dfrac{50}{A(50 - A)} = \dfrac{1}{A} + \dfrac{1}{50 - A}$.

(1 mark)

This sample Specialist Mathematics paper shows the format of the examination from November 2020.

(iii) Use integration to solve the differential equation

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}A\left(\frac{50 - A}{50}\right)$$

with initial condition $A(0) = 1$, and show that the area covered by alpha bacteria can be modelled by the equation below.

$$A = \frac{50}{1 + 49e^{-0.5t}}$$

(5 marks)

(iv) State the maximum area that is available for bacterial growth.

(1 mark)

(b) When $t = 1$, a different type of bacterium, called beta, was accidentally introduced to the Petri dish. Beta bacteria grow more quickly than alpha bacteria. The area, B, in the Petri dish that was covered by beta bacteria can be modelled by

$$B = \frac{50}{1 + 79e^{-5d}}$$

where d is time in days after the beta bacterium was introduced.

Figure 13 shows the graph for the area in the Petri dish covered by beta bacteria d days after the beta bacterium was introduced.

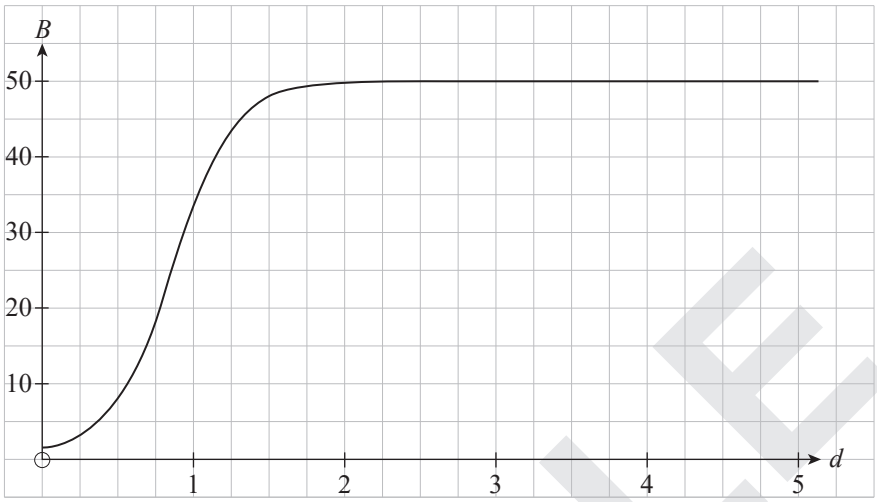


Figure 13

(i) Find d and B when the rate of growth of beta bacteria was at its greatest.

(2 marks)

(ii) Find the area of the Petri dish covered by alpha bacteria when the rate of growth of beta bacteria is at its greatest.

(2 marks)

(iii) When the Petri dish is completely covered by bacteria, which type of bacterium is likely to cover more area of the Petri dish?

(1 mark)

You may write on this page if you need more space to finish your answers to any of the questions in Question booklet 2. Make sure to label each answer carefully (e.g. 10(b)(ii) continued).