SACE BOARD
SOUTH AUSTRALIA

Stage 2 Specialist Mathematics

Sample examination questions - 2

Government of South Australia

Question 1 (7 marks)

A triangle has vertices $A(2,1,-1)$, $B(6,2,6)$, and $C(4,3,7)$.

(a) (i) Find $\overrightarrow{AC}$.

(1 mark)

(ii) Find $\left|\overrightarrow{AC}\right|$.

(1 mark)

(iii) Find $\overrightarrow{AC} \bullet \overrightarrow{AB}$.

(1 mark)

(b) Find $\angle BAC$.

(2 marks)

(c) Find $\overrightarrow{BA} \bullet \overrightarrow{BC}$.

(1 mark)

(d) Using your answers to parts (b) and (c), or otherwise, find $\angle ACB$.

(1 mark)

Question 2 (7 marks)

The shape of a garden worm may be considered to be a cylinder for the purposes of investigating its growth.

Figure 1 shows a cylinder of radius r cm and length h cm.

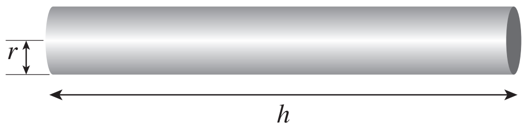

Figure 1

Garden worm

Source: © Somyot Pattana | Dreamstime.com

The growth (i.e. the rate of change of the volume) of the worm can be modelled by the rate of change of the volume of the cylinder.

(a) Show that the rate of change of the volume of the worm can be modelled by

$$\frac{dV}{dt} = \pi r\left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right)$$

where V is volume in cm^3 and t is time in weeks.

(3 marks)

(b) When the worm is 6 cm long:

- the radius of the worm, r, is 0.4 cm
- the radius of the worm is increasing at a rate of 0.05 cm per week
- the length of the worm, h, is increasing at the rate of 0.2 cm per week.

(i) Find the *exact* rate of change of the volume of the worm at the instant when $h = 6$ cm.

(2 marks)

(ii) The model shown in Figure 1 becomes more accurate when a hemisphere is added to each end of the cylinder.

Using this more accurate model, find the *exact* rate of change of the volume of the worm at the instant when $h = 6$ cm.

Note that $V_{\text{sphere}} = \frac{4}{3}\pi r^3$.

(2 marks)

Question 3 (5 marks)

(a) (i) Express $\sqrt{2} - i\sqrt{2}$ in exact polar form.

(1 mark)

(ii) Express $-\sqrt{3} + i$ in exact polar form.

(1 mark)

(b) (i) Using de Moivre's theorem, show that

$$\left(\sqrt{2} - i\sqrt{2}\right)^k \times \left(-\sqrt{3} + i\right)^3 = 2^{k+3}\operatorname{cis}\left(-\frac{k\pi}{4} + \frac{5\pi}{2}\right)$$

where k is an integer.

(2 marks)

(ii) Find the smallest positive integer value of k such that $\left(\sqrt{2}-i\sqrt{2}\right)^{k}\times\left(-\sqrt{3}+i\right)^{3}$ is purely imaginary.

(1 mark)

Question 4 (9 marks)

Consider the following set of parametric equations:

$$\begin{cases} x(t) = t^3 - 3t + 1 \\ y(t) = 1 - t \end{cases}$$

where t is a parameter and $0 \le t \le 2$.

(a) Sketch a graph of the curve defined by these parametric equations on the axes in Figure 2.

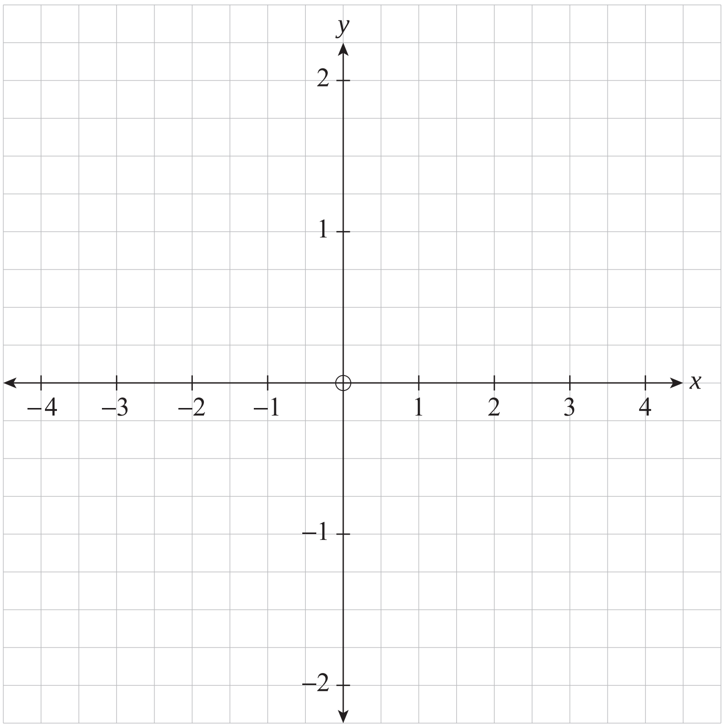

Figure 2

(3 marks)

(b) Show that the points (x, y) on the graph that you sketched in part (a) satisfy the equation

$$3y^2 - y^3 - x = 1.$$

(2 marks)

(c) Hence show that $\frac{dy}{dx} = \frac{1}{6y - 3y^2}$, where $y \neq 0$ and $y \neq 2$.

(2 marks)

(d) Hence find the slope of the tangent to the curve at the point where $t = \frac{1}{2}$.

(2 marks)

Question 5 (7 marks)

Let $f(x) = x(x-3)^5$.

(a) On the axes in Figure 3, sketch the graph of $y = f(x)$.

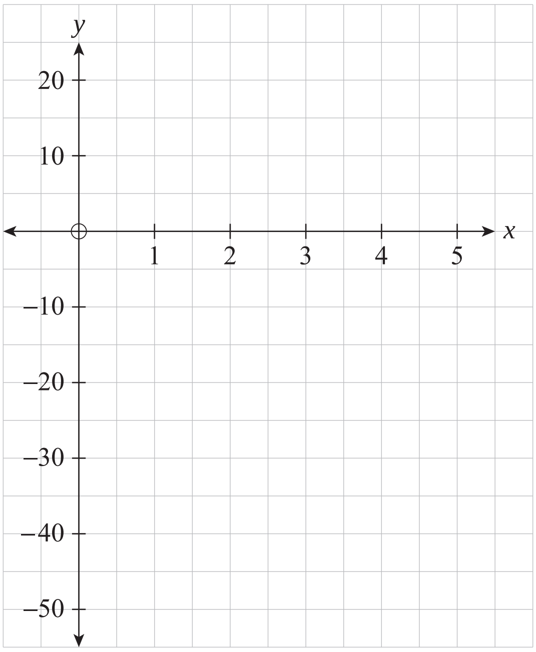

Figure 3 (2 marks)

(b) (i) Use integration by parts to show that

$$\int f(x)\,\mathrm{d}x = \frac{x}{6}(x-3)^6 - \frac{1}{42}(x-3)^7 + c$$

where c is a constant.

(3 marks)

(ii) Hence show that $\int_0^3 f(x)\,\mathrm{d}x = -\frac{729}{14}$.

(1 mark)

(iii) State the area enclosed by the x-axis and the graph of $y = f(x)$.

(1 mark)

Question 6 (7 marks)

Let $z_1 = \text{cis}\,\theta$, as shown on the Argand diagram in Figure 4.

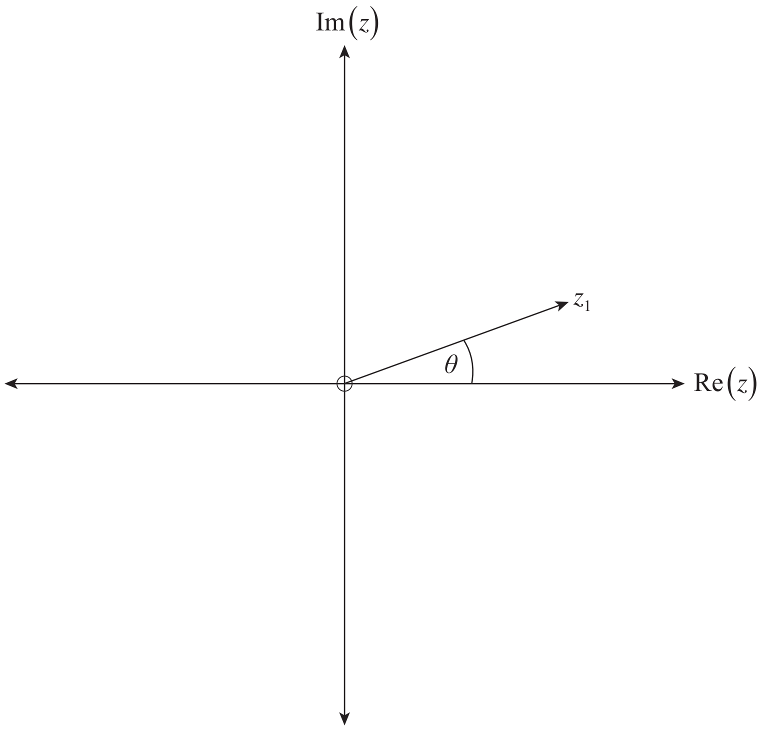

Figure 4

(a) On the Argand diagram in Figure 4:

(i) draw $z_2 = \sqrt{2}\,\text{cis}\left(\theta + \frac{\pi}{4}\right)$. (2 marks)

(ii) indicate the length $|z_1 - z_2|$. (1 mark)

(b) Use the triangle inequality to explain why $|z_1 - z_2| < 1 + \sqrt{2}$.

(2 marks)

(c) Find the exact value of $|z_1 - z_2|$.

(2 marks)

Question 7 (8 marks)

Consider the following polynomial:

$$P(x)=(x+2)\left(kx^3+ax^2+bx+c\right)+(-7x+22)$$

where k, a, b, and c are real constants.

The polynomial $P(x)$ has a factor $(x+1)$ and a zero $x=2$.

When $P(x)$ is divided by $(x-3)$, the remainder is 56.

(a) Show that an augmented matrix for the coefficients of a, b, and c can be written as

$$\begin{bmatrix} 1 & -1 & 1 & : & (k-29) \\ 9 & 3 & 1 & : & (11-27k) \\ 4 & 2 & 1 & : & (-2-8k) \end{bmatrix}.$$

(3 marks)

(b) Clearly stating all row operations, show that

$$a = 1 - 4k$$
$$b = k + 8$$
$$c = 6k - 22.$$

(4 marks)

(c) Hence or otherwise, find the polynomial $P(x)$ for $k = 2$. There is no need to simplify your answer.

(1 mark)

Question 8 (6 marks)

(a) Find the values of A and B such that $\dfrac{A}{x-3}-\dfrac{B}{(x+3)(x-3)}=\dfrac{4x}{x^2-9}$.

(2 marks)

(b) Hence or otherwise, find $\displaystyle\int \frac{12}{(x+3)(x-3)}\,\mathrm{d}x$.

(4 marks)

Question 9 (10 marks)

(a) (i) Write an expression for $\sin^2 x$ in terms of $\cos 2x$.

(1 mark)

(ii) Hence find $\int \sin^2 x \, dx$.

(2 marks)

(b) On the set of axes in Figure 5, sketch the graph of $f(x) = |\sin x| + 1$ for the interval $0 \leq x \leq 4\pi$.

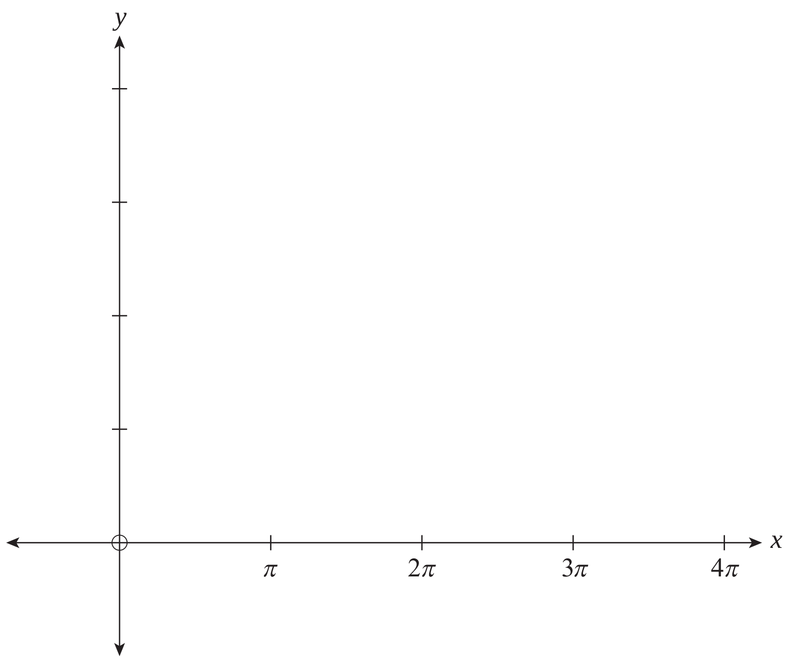

Figure 5 (3 marks)

Let $g(x) = |\sin x| + 1$ where $x \geq 0$.

An artist wishes to create a solid three-dimensional metal sculpture in the same shape as that formed when the graph of $g(x)$ is rotated around the x-axis.

(c) Consider a section of this sculpture that is represented by the region of the graph of $g(x)$ that is bounded by the lines $x = 0$ and $x = \pi$, and rotated about the x-axis.

(i) Calculate the *exact* volume of metal needed to create this section of the sculpture.

(3 marks)

(ii) If the artist has 600 cm^3 of metal, how many complete sections will her sculpture contain?

(1 mark)

Question 10 (9 marks)

(a) Simplify the following complex number: $1+\frac{i}{2}+\frac{i^2}{4}+\frac{i^3}{8}$.

(1 mark)

(b) Use mathematical induction to prove that, for all positive integers n,

$$1+\frac{i}{2}+\left(\frac{i}{2}\right)^2+\left(\frac{i}{2}\right)^3+\ldots+\left(\frac{i}{2}\right)^n=\frac{2^{n+1}-i^{n+1}}{2^n(2-i)}.$$

(5 marks)

(c) Consider $S_{20} = 1 + \frac{i}{2} + \left(\frac{i}{2}\right)^2 + \ldots + \left(\frac{i}{2}\right)^{20}$.

(i) Show that $S_{20} = \frac{\left(2^{22}+1\right)+\left(2^{21}-2\right)i}{5 \times 2^{20}}$.

(2 marks)

(ii) Write S_{20} in the form $a + bi$, correct to two decimal places.

(1 mark)

Question 11 (14 marks)

A disease is infecting a population of animals. The rate at which healthy animals become infected by the disease can be modelled by the differential equation

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -4\left(\frac{H-D}{H}\right) - \frac{1}{10}D$$

where H is the number of healthy animals, D is the initial number of infected animals, and t is time measured in days.

Suppose the initial number of infected animals, D, is 40.

(a) Show that $\dfrac{\mathrm{d}H}{\mathrm{d}t} = \dfrac{-8(H-20)}{H}$.

(2 marks)

(b) Show that $\dfrac{H}{H-20} = 1 + \dfrac{20}{H-20}$.

(1 mark)

(c) (i) Using integration, solve the differential equation in part (a) to show that

$$H + 20\ln\left|H-20\right| = -8t + c$$

where c is a constant.

(3 marks)

(ii) Given $H + 20\ln|H - 20| = -8t + c$, find the value of c, correct to three significant figures, if at time $t = 0$ there are 90 healthy animals (i.e. $H = 90$).

(2 marks)

(d) (i) Figure 6 shows the slope field for the differential equation in part (a).

Draw the solution curve if at time $t = 0$ there are 90 healthy animals.

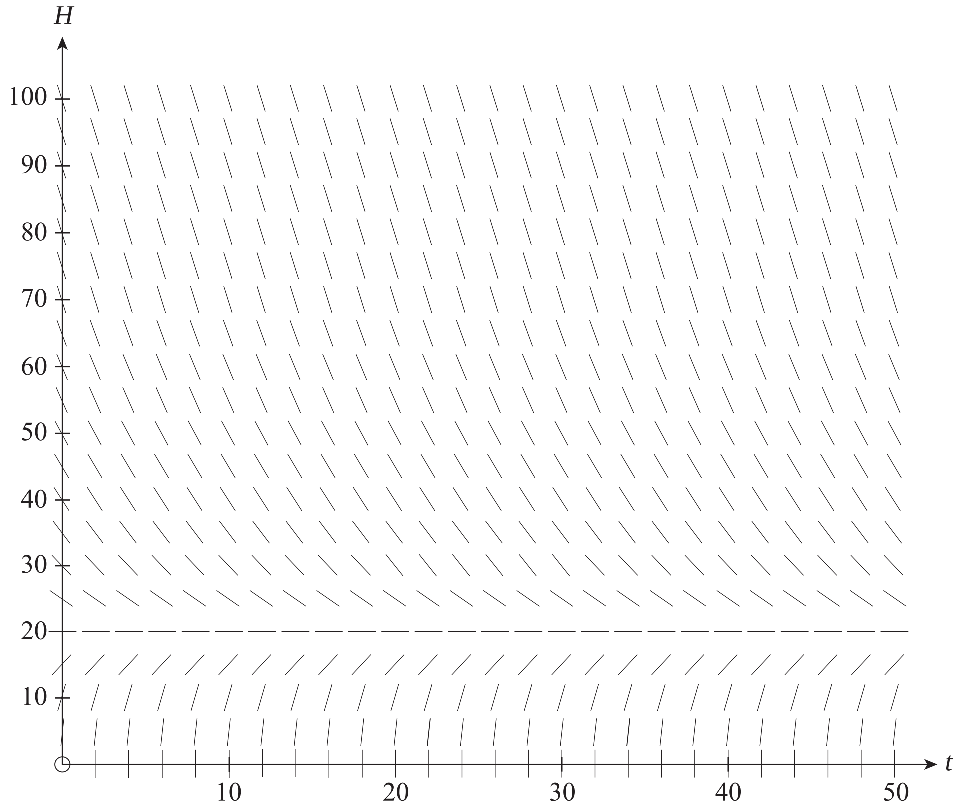

Figure 6

(3 marks)

(ii) State the limiting value of the number of healthy animals.

(1 mark)

(e) State whether there is any change to the limiting value of the number of healthy animals if at time $t = 0$ there are only 60 healthy animals.

(1 mark)

(f) At what time is the magnitude of the rate of change in the number of healthy animals greatest?

(1 mark)

Question 12 (15 marks)

The points $A(5,\ 3,\ -1)$ and $B(7,\ 5,\ 2)$ are on the line l_1, and the points $C(4,\ 1,\ 1)$, $D(6,\ 2,\ 3)$, and $E(4,\ -2,\ -2)$ are on the plane P, as shown in Figure 7.

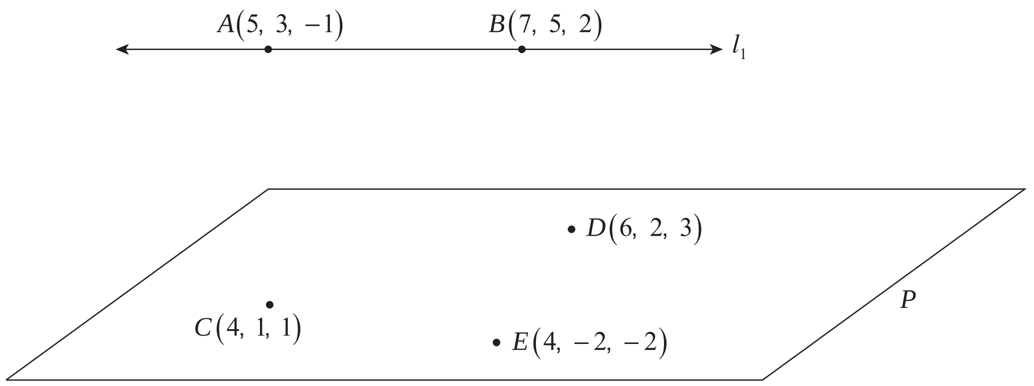

Figure 7

(a) (i) Find $\overrightarrow{CD} \times \overrightarrow{CE}$.

(2 marks)

(ii) Show that the equation of P is $x + 2y - 2z = 4$.

(2 marks)

(b) (i) Show that the parametric equations of l_1 are

$$\begin{cases} x = 5 + 2t \\ y = 3 + 2t \\ z = -1 + 3t \end{cases}$$ where t is a parameter.

(1 mark)

(ii) Show that l_1 is parallel to P.

(2 marks)

(iii) Find the distance between l_1 and P.

(2 marks)

(c) (i) Find the equation of the normal to P that intersects l_1 at A.

(1 mark)

(ii) The equation of P is $x+2y-2z=4$.

Show that the normal found in part (c)(i) passes through P at $C(4,\ 1,\ 1)$.

(1 mark)

(iii) Figure 8 shows the point F, which lies on the normal to P that passes through A and C.

The point F is 6 units from C, on the other side of P from A.

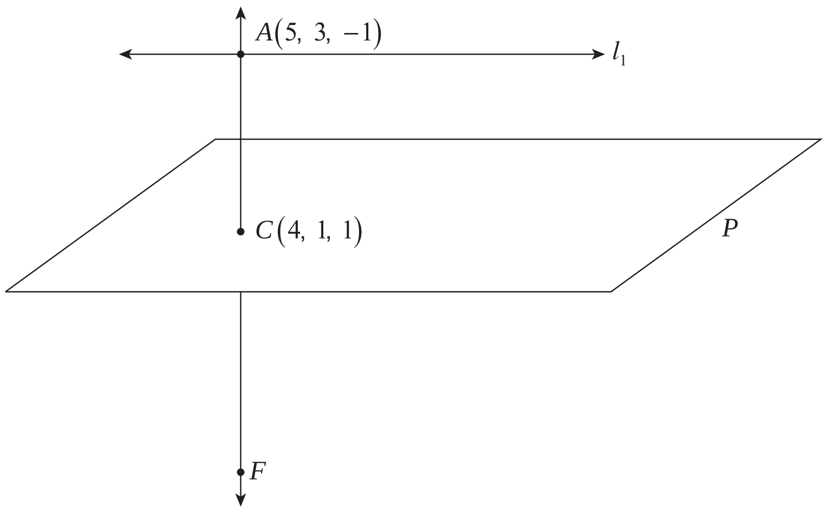

Figure 8

Find the coordinates of F.

(2 marks)

(d) Figure 9 shows the line l_2, which is parallel to l_1 and lies on the other side of P, 9 units from l_1.

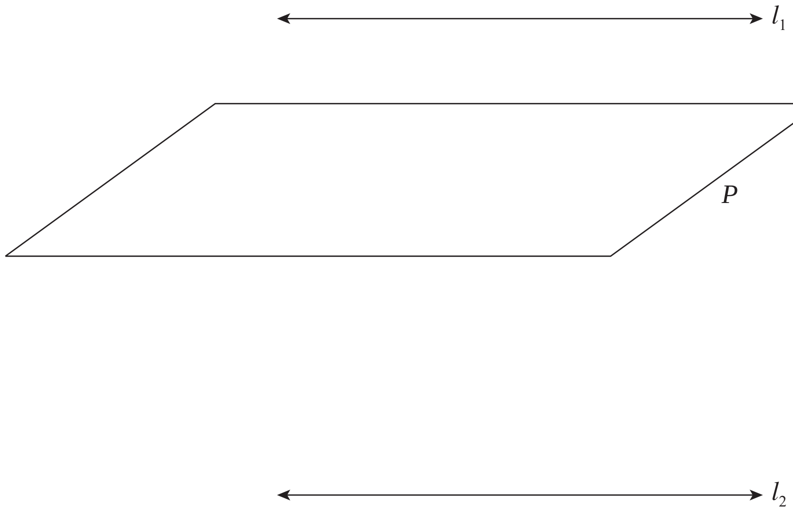

Figure 9

Find the equation of l_2.

(2 marks)

Question 13 (15 marks)

(a) (i) A real quadratic has a zero $z = 2\text{cis}\left(\frac{\pi}{3}\right)$.

State the other zero.

(1 mark)

(ii) Hence show that $z^2 - 2z + 4$ is a real quadratic that has these zeros.

(1 mark)

(b) Show that the solutions to $z^5 = -32$ are: $2\text{cis}\left(\frac{\pi}{5}\right)$; $2\text{cis}\left(\frac{3\pi}{5}\right)$; -2; $2\text{cis}\left(-\frac{3\pi}{5}\right)$; and $2\text{cis}\left(-\frac{\pi}{5}\right)$.

(3 marks)

Let $p(z)=\dfrac{(z^2-2z+4)(z^5+32)}{z+2}$, where $z\neq -2$.

(c) Show that $p(z)=z^6-4z^5+12z^4-24z^3+48z^2-64z+64$.

(2 marks)

(d) Use your answers to parts (a), (b), and (c) to solve the equation $p(z)=0$.

Write your answers exactly in $r\operatorname{cis}\theta$ form.

(2 marks)

(e) On the Argand diagram in Figure 10, plot your solutions from part (d), labelling them $z_1, z_2, \ldots z_6$ anticlockwise from the smallest positive argument.

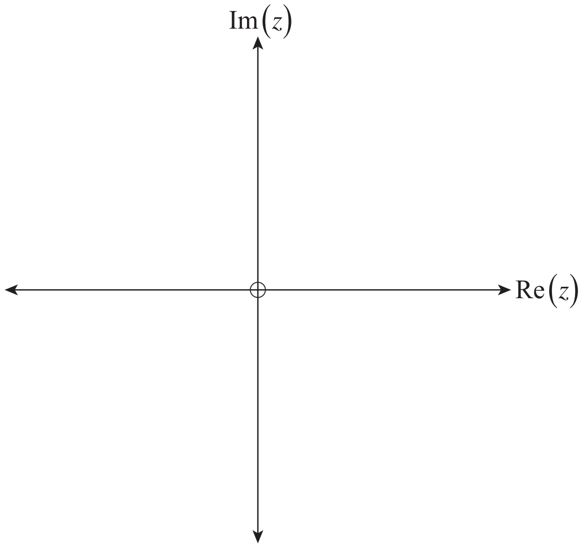

Figure 10 (2 marks)

(f) (i) Find the value of $z_1 + z_2 + z_3 + z_4 + z_5 + z_6$.

(1 mark)

(ii) Find the value of $z_1 z_2 z_3 z_4 z_5 z_6$.

(1 mark)

(g) If z_α and z_β are any two distinct solutions to the equation $p(z) = 0$, find the minimum value of $|z_\alpha - z_\beta|$.

(2 marks)

Question 14 (15 marks)

Figure 11 shows the graph of $g(x) = 2\tan x$, where $-\pi \le x \le \pi$, $x \ne \pm\frac{\pi}{2}$.

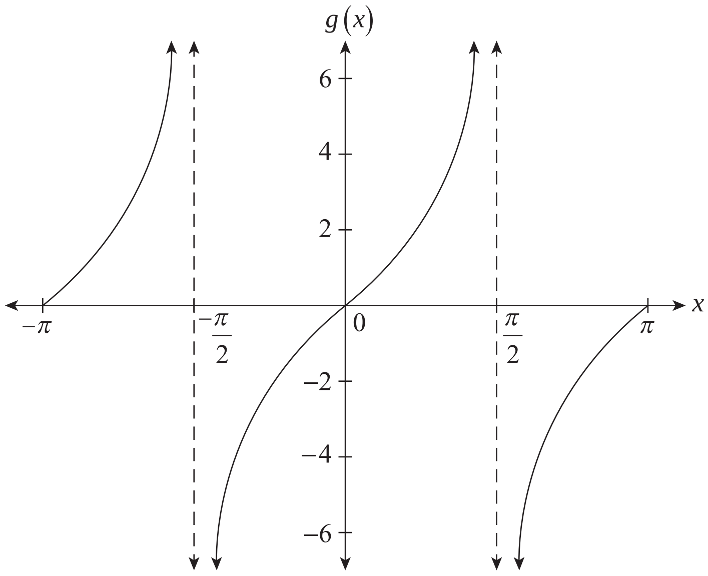

Figure 11

(a) Explain why $g(x)$ is a function but does not have an inverse function.

(2 marks)

(b) (i) Explain why the following function *does* have an inverse function:

$$f(x) = 2\tan x, \text{ where } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

(1 mark)

(ii) Show that $f^{-1}(x) = \arctan \frac{x}{2}$.

(2 marks)

(iii) On the axes in Figure 12, sketch $f^{-1}(x) = \arctan \frac{x}{2}$.

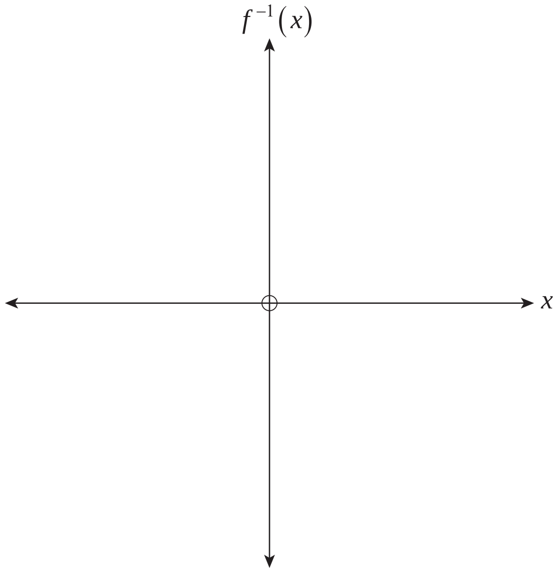

Figure 12

(2 marks)

(iv) State the domain and range of $f^{-1}(x)$ in exact form.

(2 marks)

(c) (i) If $y = \arctan\frac{x}{2}$, then $\frac{x}{2} = \tan y$.

Hence use implicit differentiation to show that $\frac{dy}{dx} = \frac{2}{4+x^2}$.

(3 marks)

(ii) Hence or otherwise, use integration to find the *exact* value of $\int_{-2}^{2} \frac{1}{4+x^2}\,dx$.

(3 marks)

Question 15 (16 marks)

The following parametric equations describe a curve:

$$\begin{cases} x = a\cos t - \cos(at) \\ y = a\sin t - \sin(at) \end{cases}$$

where t is a parameter, and $0 \le t \le 2\pi$.

Figure 13 shows the graph of the curve defined by these parametric equations where $a = 3$.

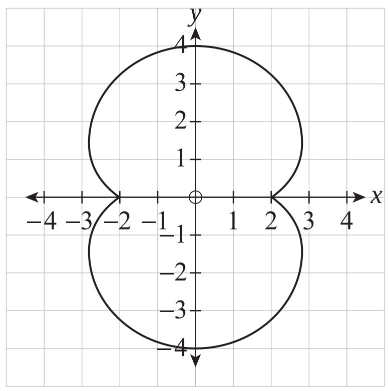

Figure 13

Now consider the case where $a = 4$.

(a) (i) Find the parametric equations where $a = 4$.

(1 mark)

(ii) On the axes in Figure 14, sketch the curve defined by these parametric equations.

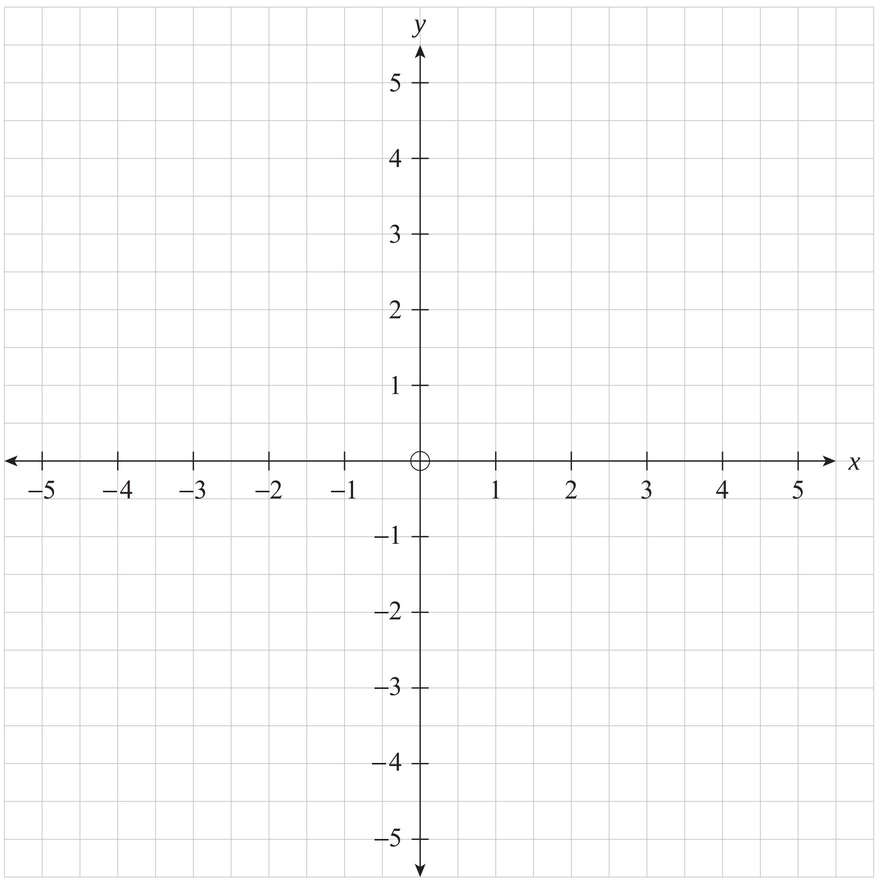

Figure 14

(3 marks)

(iii) Find the *exact* coordinates of the point for which $t = \frac{2\pi}{3}$.

(1 mark)

(iv) On the curve you sketched on Figure 14, label this point 'A'. (1 mark)

(b) (i) Assuming $\frac{dy}{dx}$ exists, find an expression for $\frac{dy}{dx}$ in terms of t.

(2 marks)

(ii) Determine the *exact* value of the slope of the tangent to the curve at the point where $t = \frac{\pi}{3}$.

(1 mark)

(c) (i) Show that the expression for the length of the curve from $t=0$ to $t=\frac{2\pi}{3}$ can be written as

$$\int_0^{\frac{2\pi}{3}} \sqrt{32-32\cos 3t}\ \mathrm{d}t.$$

(3 marks)

(ii) Show that $32-32\cos 3t = 64\sin^2\left(\frac{3t}{2}\right)$.

(1 mark)

(iii) Hence determine the *exact* value of $\int_0^{\frac{2\pi}{3}} \sqrt{32-32\cos 3t}\ \mathrm{d}t$.

(2 marks)

(d) State the length of the curve that you sketched on Figure 14.

(1 mark)